$$= \int_0^{\pi/2} \frac{x}{1+2\cos^2\frac{x}{2}-1}dx + \int_0^{\pi/2} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}dx$$

$$= \frac{1}{2}\int_0^{\pi/2} x\sec^2\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{1}{2}\left[\left(2x\tan\frac{x}{2}\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\tan\frac{x}{2}dx\right] + -\int_0^{\frac{\pi}{2}}\tan\frac{x}{2}dx$$

$$= \left(x\tan\frac{x}{2}\right)_0^{\pi/2}$$

$$= \frac{\pi}{2}\tan\frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Q. 4. Evaluate :

$\int_0^{\pi/4} \log(1+\tan x)\,dx$ *(CBSE Outside Delhi, 2011)*

Solution :

$$I = \int_0^{\pi/4}\log(1+\tan x)dx \quad ...(1)$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log\left(1+\left\{\frac{\tan\frac{\pi}{4}-\tan x}{1-\tan\frac{\pi}{4}\tan x}\right\}\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log 2\,dx - \int_0^{\pi/4}\log(1+\tan x)\,dx$$

$$\Rightarrow \quad I = (\log 2)\,[x]_0^{\pi/4} - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2$$

$$\therefore \quad I = \frac{\pi}{8}\log 2 \quad \text{[from (1)]}$$

9 AREA OF BOUNDED REGIONS

IMPORTANT FORMULAE

1. Area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$ is given by :

$$\int_a^b y\,dx = \int_a^b f(x)dx$$

2. Area of the region bounded by the curve $x = \phi\, f(y)$, y-axis and the lines $y = c$, $y = d$ $(d > c)$ is given by :

$$\int_c^d x\,dx = \int_c^d \phi(y)\,dy$$

3. Area of the region bounded by the two curves $y = f(x)$, $y = g(x)$ and lines $x = a$, $x = b$ is given by :

$$= \int_a^b [f(x) - g(x)]dx, \text{ where in } [a, b], f(x) \geq g(x)$$

4. If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then :

$$\text{Area} = \int_a^c [f(x)-g(x)]dx + \int_c^b [g(x)-f(x)]dx$$

Multiple Choice Questions

1. The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$, $0 \leq x \leq \frac{\pi}{2}$ is

(a) $\sqrt{2}$ (b) $\sqrt{2}+1$ (c) $\sqrt{2}-1$ (d) $2(\sqrt{2}-1)$

2. If the area bounded by the curves $y^2 = 4ax$ and $y = m\lambda$ is $\frac{a^2}{3}$, then the value of m is

(a) 2 (b) – 2

(c) $\frac{1}{2}$ (d) none of these

3. The area enclosed within the curve $|x| + |y| = 1$ is

(a) 21 (b) 1.5

(c) 2 (d) none of these

4. The area of the quadrilateral formed by the lines $y = 2x + 3$, $y = 0$, $x = 4$, $x = 6$ is

(a) 26 square unit (b) 20 square unit

(c) 24 square unit (d) none of these

5. Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

(a) $-a$ (b) $-\frac{15}{4}$ (c) $\frac{15}{4}$ (d) $\frac{17}{4}$

6. If the area between $x = y^2$ and $x = 4$ is divided into two square parts by the line $x = a$, then the value of a is

(a) $2^{2/3}$ (b) $4^{2/3}$ (c) $2^{3/2}$ (d) 4

7. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

8. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

(a) 2 (b) $\frac{9}{4}$ (c) $\frac{9}{2}$ (d) 3

9. The area bounded by the curve $y = x\,|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

$$= \int_0^{\pi/2} \frac{x}{1+2\cos^2\frac{x}{2}-1}dx + \int_0^{\pi/2} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}dx$$

$$= \frac{1}{2}\int_0^{\pi/2} x\sec^2\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{1}{2}\left[\left(2x\tan\frac{x}{2}\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\tan\frac{x}{2}dx\right] + -\int_0^{\frac{\pi}{2}}\tan\frac{x}{2}dx$$

$$= \left(x\tan\frac{x}{2}\right)_0^{\pi/2}$$

$$= \frac{\pi}{2}\tan\frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Q. 4. Evaluate :

$\int_0^{\pi/4}\log(1+\tan x)\,dx$ *(CBSE Outside Delhi, 2011)*

Solution :

$$I = \int_0^{\pi/4}\log(1+\tan x)dx \quad \ldots(1)$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log\left(1+\left\{\frac{\tan\frac{\pi}{4}-\tan x}{1-\tan\frac{\pi}{4}\tan x}\right\}\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4}\log 2\,dx - \int_0^{\pi/4}\log(1+\tan x)\,dx$$

$$\Rightarrow \quad I = (\log 2)\,[x]_0^{\pi/4} - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2$$

$$\therefore \quad I = \frac{\pi}{8}\log 2 \quad \text{[from (1)]}$$

9 AREA OF BOUNDED REGIONS

IMPORTANT FORMULAE

1. Area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$ is given by :

   $\int_a^b y\,dx = \int_a^b f(x)dx$

2. Area of the region bounded by the curve $x = \phi\, f(y)$, y-axis and the lines $y = c, y = d$ $(d > c)$ is given by :

   $\int_c^d x\,dx = \int_c^d \phi(y)\,dy$

3. Area of the region bounded by the two curves $y = f(x)$, $y = g(x)$ and lines $x = a, x = b$ is given by :

   $= \int_a^b [f(x) - g(x)]dx$, where in $[a, b], f(x) \geq g(x)$

4. If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then :

   $\text{Area} = \int_a^c [f(x)-g(x)]dx + \int_c^b [g(x)-f(x)]dx$

Multiple Choice Questions

1. The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$, $0 \leq x \leq \frac{\pi}{2}$ is

   (a) $\sqrt{2}$ (b) $\sqrt{2}+1$ (c) $\sqrt{2}-1$ (d) $2(\sqrt{2}-1)$

2. If the area bounded by the curves $y^2 = 4ax$ and $y = m\lambda$ is $\frac{a^2}{3}$, then the value of m is

   (a) 2 (b) – 2
   (c) $\frac{1}{2}$ (d) none of these

3. The area enclosed within the curve $|x| + |y| = 1$ is

   (a) 21 (b) 1.5
   (c) 2 (d) none of these

4. The area of the quadrilateral formed by the lines $y = 2x + 3, y = 0, x = 4, x = 6$ is

   (a) 26 square unit (b) 20 square unit
   (c) 24 square unit (d) none of these

5. Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

   (a) $-a$ (b) $-\frac{15}{4}$ (c) $\frac{15}{4}$ (d) $\frac{17}{4}$

6. If the area between $x = y^2$ and $x = 4$ is divided into two square parts by the line $x = a$, then the value of a is

   (a) $2^{2/3}$ (b) $4^{2/3}$ (c) $2^{3/2}$ (d) 4

7. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

   (a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

8. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

   (a) 2 (b) $\frac{9}{4}$ (c) $\frac{9}{2}$ (d) 3

9. The area bounded by the curve $y = x\,|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is

   (a) 0 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

10. Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

Ans. 1. (c), 2. (a), 3. (c), 4. (a), 5. (d), 6. (b), 7. (a), 8. (b), 9. (c), 10. (b).

➠ Very Short Answer Type Questions

Q. 1. Find the area of region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$. *(BSER, 2013)*

Solution :

$$\text{Required Area} = \int_{-1}^{1}(3x+2)\,dx$$

$$= 3\int_{-1}^{1} x\,dx + 2\int_{-1}^{1} 1\,dx = 3\left[\frac{x^2}{2}\right]_{-1}^{1} + 2[x]_{-1}^{1}$$

$$= 3\left[\frac{1}{2}-\frac{1}{2}\right] + 2\,[1+1]$$

$$= 3 \times 0 + 4 = 4 \text{ square unit}$$

Q. 2. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$. *(BSER, 2014)*

Solution :

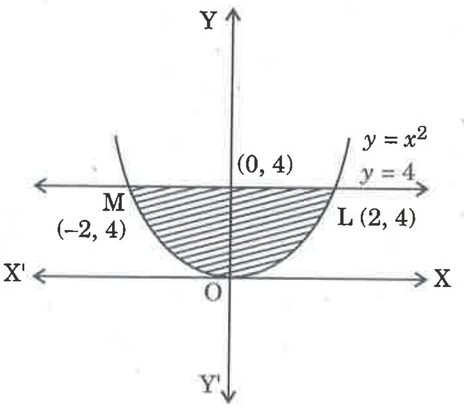
$$y = x^2 \quad \ldots(1)$$

$$y = 4 \quad \ldots(2)$$

Solving equations (1) and (2), we get

$$x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$



∴ The points of intersection are (2, 4) and (– 2, 4).

$$\text{Required Area} = 2\int_0^4 \sqrt{y}\,dy$$

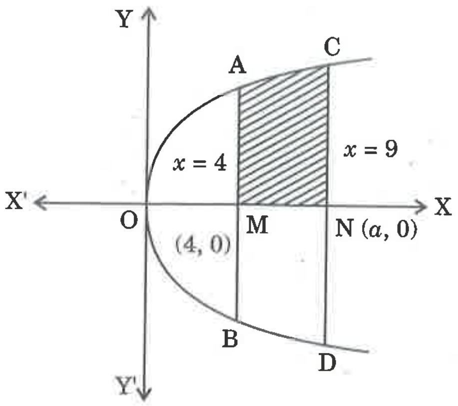
$$= 2 \times \frac{2}{3}\left(y^{\frac{3}{2}}\right)_0^4 = \frac{4}{3}4^{3/2}$$

$$= \frac{4}{3}(2^2)^{3/2} = \frac{4}{3}.8$$

$$= \frac{32}{3} \text{ square units}$$

Q. 3. Find the area of the region bounded by the parabola $y^2 = 4\,ax$, its axis and two ordinates $x = 4$ and $x = 9$. *(JAC, 2013)*

Solution :



$$\text{Required Area} = \int_4^9 2\sqrt{ax}\,dx$$

$$= 2\sqrt{a}\,\frac{2}{3}[x^{3/2}]_4^9$$

$$= \frac{4\sqrt{a}}{3}(9^{3/2} - 4^{3/2})$$

$$= \frac{4\sqrt{a}}{3}(27-8)$$

$$= \frac{76\sqrt{a}}{3} \text{ square units}$$

Q. 4. By using integration method, find the area of the circle $x^2 + y^2 = a^2$. *(USEB, 2013)*

Solution :

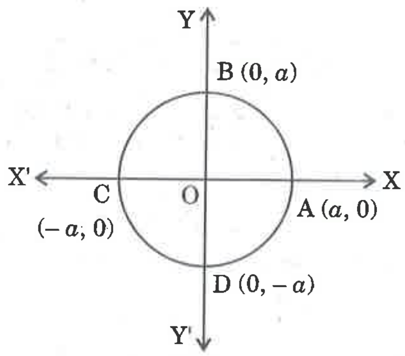


Fig.

$$\text{Area of the circle} = 4\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= 4\left[\frac{1}{2}\times\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 2\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 2\,[a^2\sin^{-1}(1)]$$

$$= 2a^2\frac{\pi}{2}$$

$$= \pi a^2 \text{ square units}$$

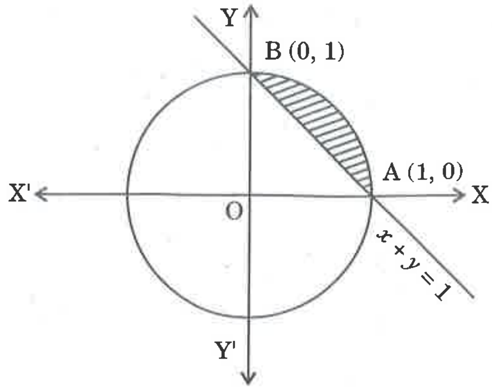
➠ Short Answer Type Questions

Q. 1. Find the area of the region given by $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$ *(BSEB, 2014)*

Solution :

Required Area = Area of Quadrant OAB

– Area of ΔOAB



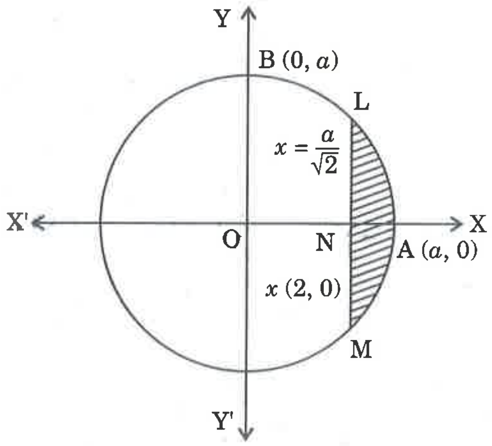
$$= \int_0^1 \sqrt{1-x^2}\,dx - \frac{OA.OB}{2}$$

$$= \left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 - \frac{|x|}{2}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\right) \text{ square units}$$

Q. 2. Find the area of smaller part bounded by the circle $x^2 + y^2 = a^2$ and the line $x = \frac{a}{\sqrt{2}}$. (*BSER*, 2013)

Solution :



Area of the smaller part

$$= 2\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx \text{ (Due to symmetry)}$$

$$= 2\left[\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_{a/\sqrt{2}}^{a}$$

$$= \left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_{a/\sqrt{2}}^{a}$$

$$= \left[a\sqrt{a^2-a^2} + a^2\sin^{-1}(1)\right]$$

$$-\left[\frac{a}{\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} + a^2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= a^2\frac{\pi}{2} - \left[\frac{a}{\sqrt{2}}.\frac{a}{\sqrt{2}} + a^2\frac{\pi}{4}\right]$$

$$= a^2\frac{\pi}{2} - \frac{a^2}{2}.\frac{a^2\pi}{4}$$

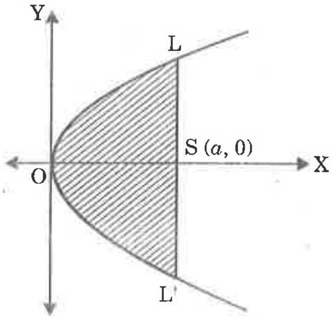
$$= \frac{a^2\pi}{4} - \frac{a^2}{2}$$

$$= a^2\left(\frac{\pi}{4} - \frac{1}{2}\right) \text{ square units}$$

Q. 3. Calculate the area bounded by the parabola $y^2 = 4ax$ and its latus rectum. (*USEB*, 2010)

Solution :

The equation of the parabola is $y^2 = 4ax$ ...(i)



Here the focus S is $(a, 0)$, let latus rectum is LSL' which is a line parallel to y-axis at a distance of a from it so its equation is $x = a$.

Since all the power of y in the curve (i) are even, so the curve will be symmetrical about x-axis.

∴ Required Area

= area LOL'L

= 2 × area LOSL

$$= 2\int_0^a y\,dx = 2\int_0^a 2\sqrt{ax}\,dx$$

$$= 4\sqrt{a}\,\frac{2}{3}[x^{3/2}]_0^a = \frac{8}{3}a^2 \text{ sq. units.}$$

Q. 4. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$. (*AI CBSE*, 2013)

Solution :

The given curves are

$y = x^2$ ...(1)

and $y = |x|$ ...(2)

(1) and (2) give

$x^2 = |x|$

Case I When $x \le 0$

$x^2 = -x$

⇒ $x(x+1) = 0$

⇒ $x = 0, -1$

∴ From (1), $y = 0, 1$

Case II when $x \ge 0$

$x^2 = x$

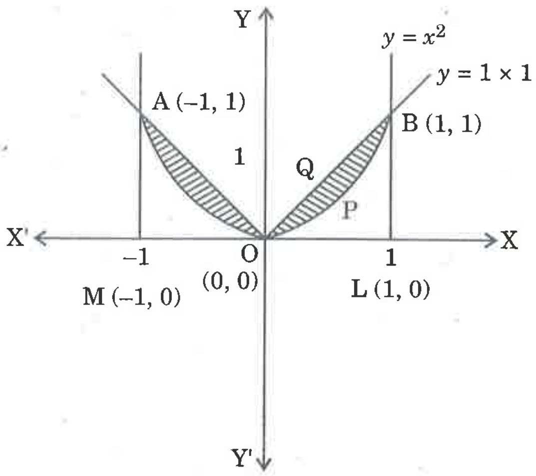
⇒ $x(x-1) = 0$

⇒ $x = 0, 1$

∴ From (1),

$y = 0,1$

Hence, (1) and (2) intersect at the points O (0, 0), A (– 1, 1) and B (1, 1).



Required Area $= 2$ Area OPBO (By symmetry)

$= 2$ [Area OQBLO – Area OPBLO]

$$= 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right]$$

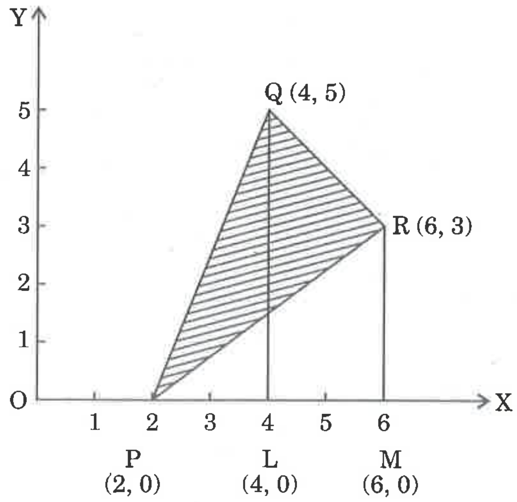
$$= 2\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1\right]$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3} \text{ square unit}$$

Long Answer Type Questions

Q. 1. Using integration, find the area of the triangle PQR, co-ordinates of whose vertices are P(2, 0), Q (4, 5) and R (6, 3). (*AI, CBSE*, 2014 (*Comptt.*))

Solution :



We have P ⟶ (2, 0)

Q ⟶ (4, 5)

R ⟶ (6, 3)

Equation of line PQ is

$$y - 0 = \frac{5-0}{4-2}(x-2)$$

$\Rightarrow$ $2y = 5x - 10$

$\Rightarrow$ $y = \frac{5}{2}x - 5$ ...(1)

Equation of line QR is

$$y - 5 = \frac{3-5}{6-4}(x-4)$$

$\Rightarrow$ $2y - 10 = -2x + 8$

$\Rightarrow$ $2y = -2x + 18$

$\Rightarrow$ $y = -x + 9$ ...(2)

Equation of line PR is

$$y - 0 = \frac{3-0}{6-2}(x-2)$$

$\Rightarrow$ $4y = 3x - 6$

$\Rightarrow$ $y = \frac{3}{4}x - \frac{3}{2}$ ...(3)

Area of the triangle PQR

= Area PQL + Area LQRM – Area PRM

$$= \int_2^4\left(\frac{5}{2}x - 5\right)dx + \int_4^6(-x+9)\,dx - \int_2^6\left(\frac{3}{4}x - \frac{3}{2}\right)dx$$

$$= \left(\frac{5x^2}{4} - 5x\right)_2^4 + \left(-\frac{x^2}{2} + 9x\right)_4^6 - \left(\frac{3x^2}{8} - \frac{3}{2}x\right)_2^6$$

$$= [(20-20)-(5-10)] + [(-18+54)-(-8+36)] - \left[\left(\frac{27}{2}-9\right)-\left(\frac{3}{2}-3\right)\right]$$

$= 5 + 8 - 6$

$= 7$ square units

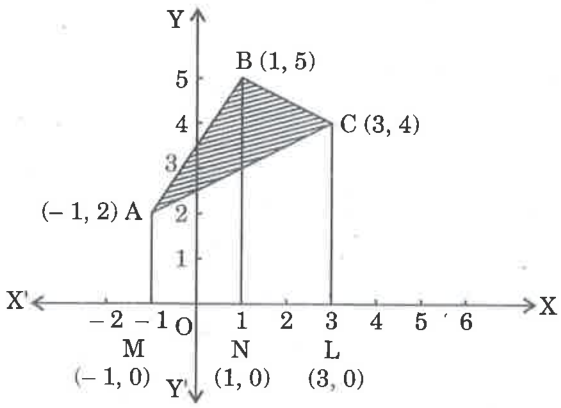
Q. 2. Using integration, find the area of the region bounded by the triangle whose vertices are (– 1, 2), (1, 5) and (3, 4). (*AI, CBSE*, 2014)

Solution :

Let A ⟶ (– 1, 2)

B ⟶ (1, 5)

and C ⟶ (3, 4)



then, equation of line AB is

$$y - 2 = \frac{5-2}{1+1}(x+1)$$

$\Rightarrow$ $2y - 4 = 3x + 3$

$\Rightarrow$ $2y - 3x = 7$ ...(1)

Equation of line BC is

$$y - 5 = \frac{4-5}{3-1}(x-1)$$

$\Rightarrow$ $2y - 10 = -x + 1$

$\Rightarrow$ $2y + x = 11$ ...(2)

Equation of line AC is

$$y - 2 = \frac{4-2}{3+1}(x+1)$$

$$\Rightarrow \quad 4y - 8 = 2x + 2$$

$$\Rightarrow \quad 4y - 2x = 10$$

$$\Rightarrow \quad 2y - x = 5 \qquad ...(3)$$

Required Area

= Area of OABC

= Area AMNB + Area BNLC – Area AMLC

$$= \int_{-1}^{1} \frac{3x+7}{2}\,dx + \int_{1}^{3} \frac{11-x}{2}\,dx - \int_{-1}^{3} \frac{x+5}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{3x^2}{2} + 7x\right)_{-1}^{1} + \frac{1}{2}\left(11x - \frac{x^2}{2}\right)_{1}^{3} - \frac{1}{2}\left(\frac{x^2}{2} + 5x\right)_{-1}^{3}$$

$$= \frac{1}{2}\left[\left(\frac{3}{2}+7\right) - \left(\frac{3}{2}-7\right)\right] + \frac{1}{2}\left[\left(33 - \frac{9}{2}\right) - \left(11 - \frac{1}{2}\right)\right]$$

$$- \frac{1}{2}\left[\left(\frac{9}{2}+15\right) - \left(\frac{1}{2}-5\right)\right]$$

= 7 + 9 – 12

= 4 square units

Q. 3. Using integration, find the area of the region bounded by the lines $2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$. [*AI, CBSE*, 2014 (Comptt.)]

Solution :

The given lines are

$$2x + y = 4 \qquad ...(1)$$

$$3x - 2y = 6 \qquad ...(2)$$

$$x - 3y + 5 = 0$$

Solving equations (1) and (2), we get

$$x = 2, y = 0$$

Solving equations (2) and (3), we get

$$x = 4, y = 3$$

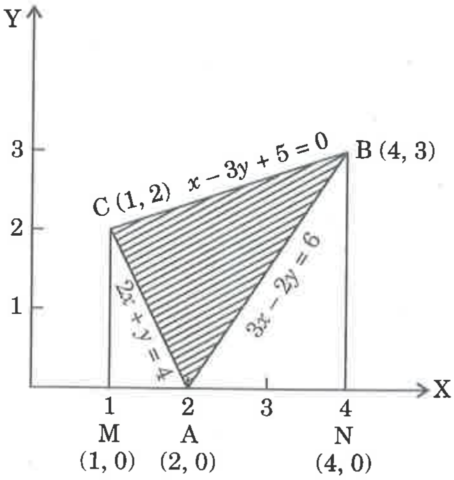
Solving equations (3) and (1), we get

$$x = 1, y = 2$$

Required Area = Area of ΔABC

= Area MCBN – [Area MCA + Area ANB]

$$= \int_{1}^{4}\left(\frac{x+5}{3}\right)dx - \int_{1}^{2}(4-2x)dx - \int_{2}^{4}\left(\frac{3x-6}{2}\right)dx$$



$$= \frac{1}{3}\left(\frac{x^2}{2} + 5x\right)_{1}^{4} - (4x - x^2)_{1}^{2} - \frac{1}{2}\left(\frac{3x^2}{2} - 6x\right)_{2}^{4}$$

$$= \frac{1}{3}\left\{(8+20) - \left(\frac{1}{2}+5\right)\right\} - \{(8-4) - (4-1)\}$$

$$- \frac{1}{2}\{(24-24) - (6-12)\}$$

$$= \frac{45}{6} - 1 - 3$$

$$= \frac{21}{6}$$

$$= \frac{7}{2} \text{ square units}$$

Q. 4. Using integration, find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$. [*CBSE*, 2013; 14 (*Comptt.*)]

Solution :

The equations of the curves are

$$x^2 = 4y \qquad ...(1)$$

and $$y = 4y - 2 \qquad ...(2)$$

Solving equations (1) and (2), we get

$$x^2 - x = 2$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad (x-2)(x+1) = 0$$

$$\Rightarrow \quad x = 2, -1$$

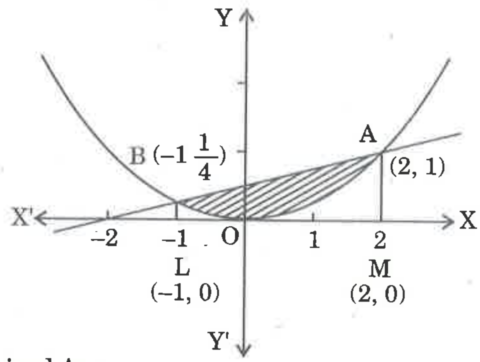
From (2),

when $x = 2, y = 1$

and when $x = -1, y = \frac{1}{4}$

Hence the curves (1) and (2) intersect at the points

A (2, 1) and B $\left(-1, \frac{1}{4}\right)$



Required Area

= Area OBAO

= Area LBAML – Area LBOAML

$$= \int_{-1}^{2} \frac{x+2}{4}\,dx - \int_{-1}^{2} \frac{x^2}{4}\,dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2} + 2x\right)_{-1}^{2} - \frac{1}{4}\left(\frac{x^3}{3}\right)_{-1}^{2}$$

$$= \frac{1}{4}\left\{(2+4) - \left(\frac{1}{2} - 2\right) - \frac{1}{4}\left(\frac{8}{3} + \frac{1}{3}\right)\right\}$$

$$= \frac{9}{8} \text{ square units}$$

Q. 5. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. *(CBSE, 2014)*

Solution :

The equations of the curves are

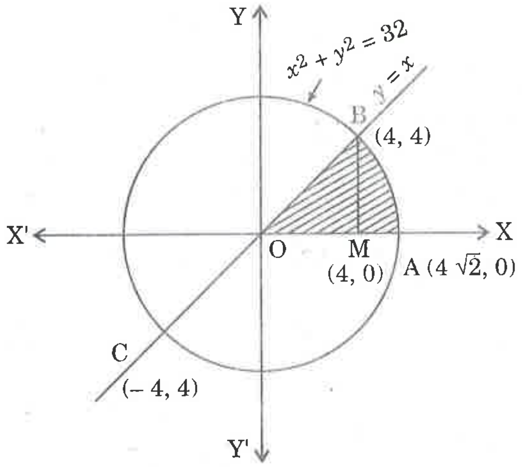
$$y = x \quad (1)$$

and $$x^2 + y^2 = 32 \quad ...(2)$$

Solving equations (1) and (2), we get

$$x = \pm 4, y = \pm 4$$

Hence the line (1) and the circle (2) intersect at the points B (4, 4) and C (– 4, – 4).



Required Area

$= \text{Area OABO}$

$= \text{Area OMBO} + \text{Area AMBA}$

$$= \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^4 + \left[\frac{1}{2}x\sqrt{32-x^2} + \frac{1}{2}.32.\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$

$$= 8 + 16.\frac{\pi}{2} - \left[\frac{1}{2}.4.4 + \frac{1}{2}.32.\frac{\pi}{4}\right]$$

$= 8 + 8\pi - 8 - 4\pi$

$= 4\pi$ square units

Q. 6. Find the area of the region $\{(x, y) : y^2 \le 6ax$ and $x^2 + y^2 \le 16a^2\}$ using method of integration. *(AI CBSE, 2013)*

Solution :

The given curves are

$$y^2 = 6ax \quad ...(1)$$

and $$x^2 + y^2 \le 16a^2 \quad ...(2)$$

Solving (1) and (2), we get

$$x^2 + 6ax = 16a^2$$

$\Rightarrow \quad x^2 + 6ax - 16a^2 = 0$

$\Rightarrow \quad x^2 + 8ax - 2ax - 16a^2 = 0$

$\Rightarrow \quad x(x + 8a) - 2a(x + 8a) = 0$

$\Rightarrow \quad (x + 8a)(x - 2a) = 0$

$\Rightarrow \quad x = -8a, 2a$

$\therefore$ From (1),

when $x = -8a$, $y^2 = 6a(-8a) = -48a^2$

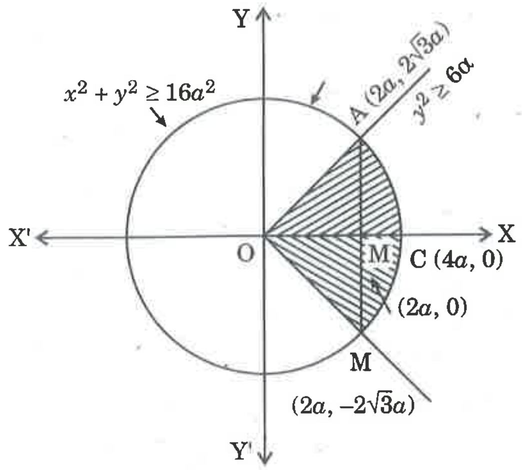
which is inadmissible as it gives imaginary values of y.

when $x = 2a$, $y^2 = 6a(2a) = 12a^2$

$\Rightarrow \quad y = \pm 2\sqrt{3}a$

Hence, the points of integration of the curves (1) and (2) are

A $(2a, 2\sqrt{3}a)$ and B $(2a, -2\sqrt{3}a)$



Required Area

$= 2 \text{ area OCAO}$ (Due to symmetry)

$= 2[\text{Area OMAO} + \text{Area AMCA}]$

$$= \left[\int_0^{2a} \sqrt{6ax}\,dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\,dx\right]$$

$$= 2\sqrt{6a}\,\frac{2}{3}\left(x^{3/2}\right)_0^{2a}$$

$$+ 2\left[\frac{1}{2}\times\sqrt{16a^2 - x^2} + \frac{1}{2}(16a^2)\sin^{-1}\left(\frac{x}{4a}\right)\right]_{2a}^{4a}$$

$$= \frac{4\sqrt{6a}}{3}2^{3/2}a^{3/2} + 2[0 + 8a^2\sin^{-1}(1)]$$

$$-2\left[\frac{1}{2}.2a.2\sqrt{3}a + 8a^2\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{16\sqrt{3}a^2}{3} + 8\pi a^2 - 4\sqrt{3}a^2 - \frac{8\pi a^2}{3}$$

$$= \frac{16\sqrt{3}a^2}{3} + \frac{16\pi a^2}{3} - 4\sqrt{3}a^2$$

$$= \left(\frac{16\sqrt{3}a^2}{3} + \frac{16\pi a^2}{3}\right) \text{ square units}$$

Q. 7. Find the area of the region $\{(x, y) : y^2 \le 4x$, $4x^2 + 4y^2 \le a\}$ using method of integration. *(AI CBSE, 2013)*

Solution :

The given curves are

$$y^2 = 4x \quad ...(1)$$

and $$4x^2 + 4y^2 = a \quad ...(2)$$

Solving (1) and (2), we get

$$4x^2 + 16x = 9$$

$\Rightarrow \quad 4x^2 + 16x - 9 = 0$

$\Rightarrow \quad 4x^2 + 18y - 2x - 9 = 0$

$\Rightarrow \quad 2x(2x + 9) - 1(2x + a) = 0$

$\Rightarrow \quad (2x + 9)(2x - 1) = 0$

$\Rightarrow \qquad x = -\frac{9}{2}, \frac{1}{2}$

$\therefore$ from (1),

when $\qquad x = -\frac{9}{1}, y^2 = -18$

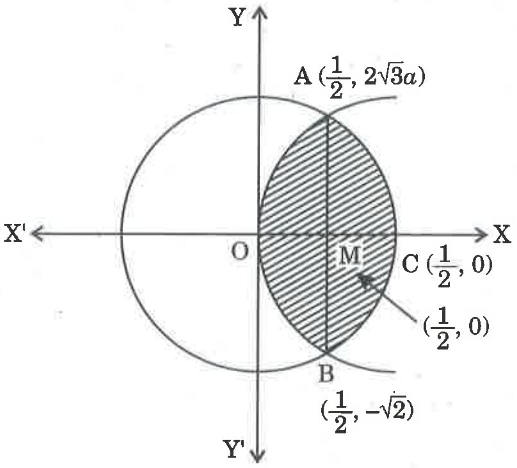
which is inadmissible it gives imaginary values of y.

and when $\quad x = \frac{1}{2}, y^2 = \sqrt{2}$

$\Rightarrow \qquad y = \pm\sqrt{2}$

Hence, the points of intersection of (1) and (2) are

A $\left(\frac{1}{2}, \sqrt{2}\right)$ and B $\left(\frac{1}{2}, -\sqrt{2}\right)$



Required Area

= Area OCAO (due to symmetry)

= 2 [Area OMAO + Area AMCA]

$$= 2\left[\int_0^{1/2}\sqrt{4x}\,dx + \int_{1/2}^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx\right]$$

$$= 2.2\int_0^{1/2}\sqrt{x}\,dx + 2\int_{1/2}^{3/2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2}\,dx$$

$$= 4.\frac{2}{3}\left(x^{3/2}\right)_0^{1/2} + 2\left[\frac{1}{2}\times\sqrt{\frac{9}{4}-x^2} + \frac{1}{2}.\frac{9}{4}\sin^{-1}\left\{\frac{x}{(3/2)}\right\}\right]_{1/2}^{3/2}$$

$$= \frac{8}{3.2\sqrt{2}} + \left[x\sqrt{\frac{9}{4}-x^2} + \frac{9}{4}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{1/2}^{3/2}$$

$$= \frac{4}{3\sqrt{2}} + \left[0 + \frac{9}{4}\sin^{-1}(1)\right] - \left[\frac{1}{2}\sqrt{2} + \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{4}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \left[\frac{4}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right] \text{ square units}$$

Q. 8. Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

[*CBSE*, 2013 (*Comptt.*)]

Solution :

The equations of the two circles are

$$x^2 + y^2 = 1 \qquad ...(1)$$

and $$(x-1)^2 + y^2 = 1 \qquad ...(2)$$

(1) is a circle with centre (0, 0) and radius 1 unit.

(2) is a circle with centre (1, 0) and radius 1 unit.

Solving equations (1) and (2), we get

$$x^2 = (x-1)^2$$

$\Rightarrow \qquad x^2 = x^2 - 2x + 1$

$\Rightarrow \qquad 2x = 1$

$\Rightarrow \qquad x = \frac{1}{2}$

$\therefore$ from (1), $y = \pm\frac{3}{\sqrt{2}}$

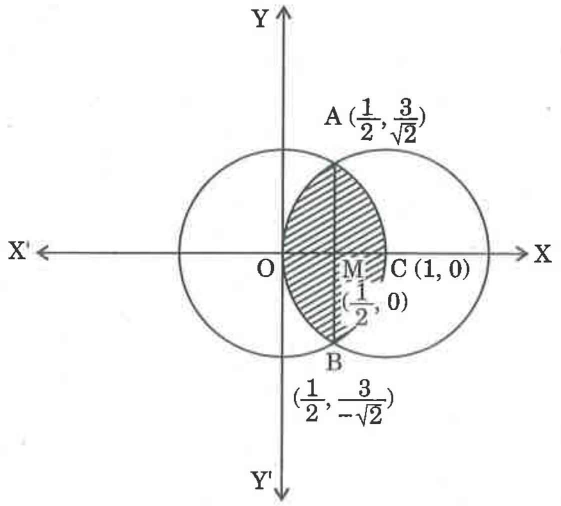
Hence, the points of intersection of (1) and (2) are

A $\left(\frac{1}{2}, \frac{3}{\sqrt{2}}\right)$ and B $\left(\frac{1}{2}, -\frac{3}{\sqrt{2}}\right)$.

Required Area

= 2 Area OACO

= 2 (Area OAMO + Area MACM)



$$= 2\left[\int_0^{1/2}\sqrt{1-(x-1)^2}\,dx + \int_{1/2}^{1}\sqrt{1-x^2}\,dx\right]$$

$$= 2\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{1/2} + 2\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{1/2}^{1}$$

$$= \left[(x-1)\sqrt{1-(x-1)^2} + \sin^{-1}(x-1)\right]_0^{1/2} + \left[x\sqrt{1-x^2} + \sin^{-1}x\right]_{1/2}^{1}$$

$$= \left[-\frac{1}{2}\frac{\sqrt{3}}{2} + \sin^{-1}\left(-\frac{1}{2}\right)\right] - \left[0 + \sin^{-1}(-1)\right] + \left[0 + \sin^{-1}(1)\right] - \left[\frac{1}{2}\frac{\sqrt{3}}{2} + \sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6}$$

$$= \left(-\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right) \text{ square units}$$

Q. 9. Find the area of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

[*CBSE*, 2013 (*Comptt.*), *CBSE*, 13]

Solution :

The equations of the two circles are

$$x^2 + y^2 = 4 \quad \ldots(1)$$

and $$(x - 2)^2 + y^2 = 4 \quad \ldots(2)$$

(1) is a circle with centre (0, 0) and radius 2 units.

(2) is a circle with centre (2, 0) and radius 2 units.

Solving equations (1) and (2), we get

$$x^2 = (x - 2)^2$$

$$\Rightarrow \quad x^2 = x^2 - 4x + 4$$

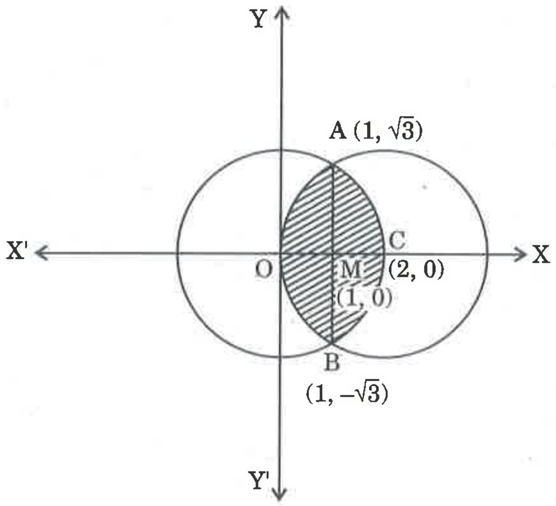
$$\Rightarrow \quad 4x = 4$$

$$\Rightarrow \quad x = 1$$

$\therefore$ from (1),

$$y = \pm \sqrt{3}$$

Hence, the points of intersection of (1) and (2) are A $(1, \sqrt{3})$ and B $(1, -\sqrt{3})$.



Required Area

= 2 Area OACO

= 2 (Area OAMO + Area MACM)

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right]$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}.4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}.4\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+\left[x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$$

$$= \left[(-1)\sqrt{3} + 4\sin^{-1}\left(-\frac{1}{2}\right)\right] - [0 + 4\sin^{-1}(-1)]$$

$$+ [0 + 4\sin^{-1}(1)] - \left[1.\sqrt{3} + 4\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= -\sqrt{3} - 4\frac{\pi}{6} + 4.\frac{\pi}{2} + 4\frac{\pi}{2} - \sqrt{3} - 4\frac{\pi}{6}$$

$$= \left(-2\sqrt{3} + \frac{8\pi}{3}\right) \text{ square units}$$

Q. 10. Using integration, find the area of the region enclosed between two circles $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$.

[*CBSE*, 2013, (*Comptt.*)]

Solution :

The equations of the two circles are

$$x^2 + y^2 = 9 \quad \ldots(1)$$

and $$(x - 3)^2 + y^2 = 9 \quad \ldots(2)$$

(1) is a circle with centre (0, 0) and radius 3 units.

(2) is a circle with centre (3, 0) and radius 3 units.

Solving equations (1) and (2), we get

$$x^2 = (x - 3)^2$$

$$\Rightarrow \quad x^2 = x^2 - 6x + 9$$

$$\Rightarrow \quad 6x = 9$$

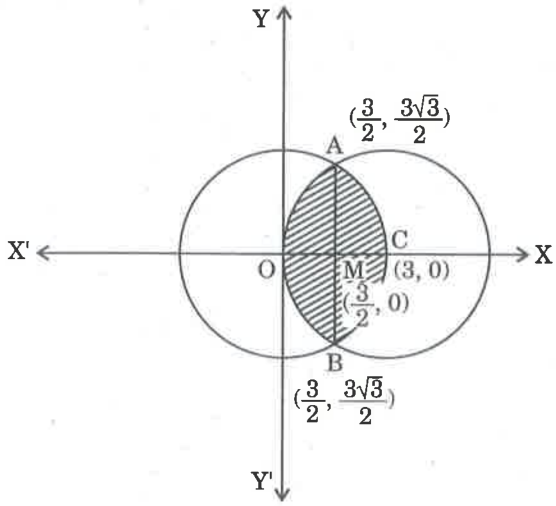
$$\Rightarrow \quad x = \frac{3}{2}$$

$\therefore$ from (1),

$$y^2 = 9 - x^2 = 9 - \frac{9}{4} = \frac{27}{4}$$

$$\Rightarrow \quad y = \pm \frac{3\sqrt{3}}{2}$$

Hence, the points of intersection of (1) and (2) are A $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ and B $\left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$.



Required Area

= 2 Area OACO

= 2 (Area OAMO + Area MACM)

$$= 2\left[\int_0^{3/2} \sqrt{9-(x-3)^2}\,dx + \int_{3/2}^3 \sqrt{9-x^2}\,dx\right]$$

$$= 2\left[\frac{1}{2}(x-3)\sqrt{9-(x-3)^2} + \frac{1}{2}.9.\sin^{-1}\left(\frac{x-3}{3}\right)\right]_0^{3/2}$$

$$+2\left[\frac{1}{2}x\sqrt{9-x^2} + \frac{1}{2}.9.\sin^{-1}\left(\frac{x}{3}\right)\right]_{3/2}^3$$

$$= \left[(x-3)\sqrt{a-(x-3)^2} + 9\sin^{-1}\left(\frac{x-3}{3}\right)\right]_0^{3/2}$$

$$+ \left[x\sqrt{9-x^2} + 9\sin^{-1}\left(\frac{x}{3}\right)\right]_{3/2}^{3}$$

$$= \left[-\frac{3}{2}.\frac{3\sqrt{3}}{2} + 9\sin^{-1}\left(-\frac{1}{2}\right)\right] - [3.0 + 9\sin^{-1}(-1)]$$

$$+ [0 + 9\sin^{-1}(1)] - \left[\frac{3}{2}.\frac{3\sqrt{3}}{2} + 9\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= -\frac{9\sqrt{3}}{4} - 9\frac{\pi}{6} + \frac{9\pi}{2} + \frac{9\pi}{2} - \frac{9\sqrt{3}}{4} - 9.\frac{\pi}{6}$$

$$= \left(-\frac{9\sqrt{3}}{2} + 6\pi\right) \text{ square units}$$

Q. 11. Find the area of the region bounded between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$. *(CBSE, Delhi, 2008)*

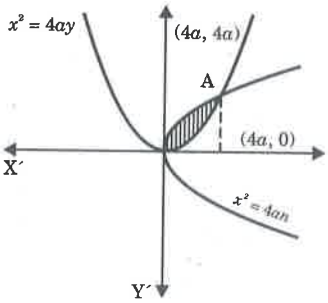
Solution :

Given $y^2 = 4ax$ ...(1)

and $x^2 = 4ay$ ...(2)

Curve (1) & (2) are both parabolas with vertices (0, 0).

Solving equations (1) and (2), we get intersect points (0, 0) and (4*a*, 4*a*).



Hence, Required Area

$$= \int_0^{4a} (y_2 - y_1)\,dx$$

$$= \int_0^{4a} \left(2\sqrt{(ax)} - \frac{x^2}{4a}\right)dx$$

From (1) & (2),

$$y = 2\sqrt{ax} \text{ and } \frac{x^2}{4a}$$

$$= \left[\left\{\frac{4}{3}\sqrt{a}x^{3/2}\right\} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{(4a)^3}{12a}$$

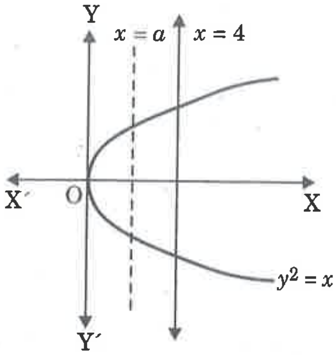
$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{ sq. units}$$

Q. 12. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$. Find the value of a.

Solution :

Given $x = y^2$ ...(1)

and $x = 4$ ...(2)



Equation (1) is parabola whose vertex is (0, 0) and eq. (2) is a line whose parallel to *y*-axis and its distance is 4 units.

Let line $x = a$, its divide in two parts, let cut the area A_1 from the line $x = 4$. Thus :

$$A_1 = 2\int_0^4 y\,dx = 2\int_0^4 x^{1/2}\,dx$$

$$= \left[\frac{4}{3}|x^{3/2}|_0^4\right] = \frac{4}{3}(2^2)^{3/2} = \frac{32}{3}$$

If cutting the area A_2 through $x = a$,

then $$A_2 = 2\int_0^a x^{1/2}dx = \frac{4}{3}a^{3/2}$$

According to question,

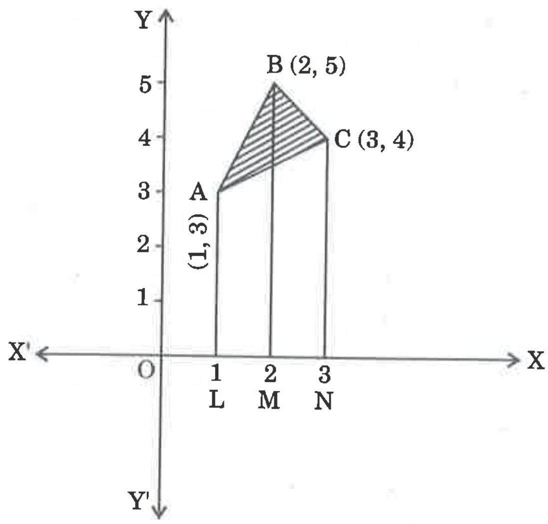
$$2A_2 = A_1$$

$$\Rightarrow \quad \frac{8}{3}a^{3/2} = \frac{32}{3}$$

$$\Rightarrow \quad a = (4)^{2/3}$$

Q. 13. Using integration, find the area of the triangle whose vertices are (1, 3), (2, 5) and (3, 4). *(JAC, 2014)*

Solution :



Equation of line AB is

$$y - 3 = \frac{5-3}{2-1}(x-1)$$

$$\Rightarrow \quad y - 3 = 2x - 2$$

$$\Rightarrow \quad y = 2x + 1 \quad ...(1)$$

Equation of line BC is

$$= 2\int_0^2 y\,dx$$

$$= 2\int_0^2 \sqrt{8x}\,dx$$

$$= 2\int_0^2 2\sqrt{2x}\,dx = 4\int_0^2 \sqrt{2x}\,dx$$

$$= 4\sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2 = 4\sqrt{2}\left[\frac{(2)^{3/2}}{3/2} - 0\right]$$

$$= 4\sqrt{2} \times \frac{2\sqrt{2}}{3} \times 2 = \frac{32}{3} \text{ square units}$$

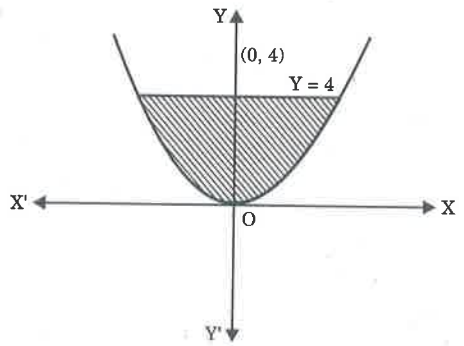
Q. 16. Find the area bounded by curve $x^2 = 4y$ and the line $y = 4$. (*USEB*, 2015)

Solution :

Given $\quad x^2 = 4y \quad$ ...(1)

and $\quad y = 4 \quad$ ...(2)

Equation (1) is parabola whose vertex is (0, 0) and equation (2) is a line which is parallel to x-axis and its distance is 4 units.



Hence, required area

$$= 2\int_0^4 x\,dy$$

$$= 2\int_0^4 \sqrt{4y}\,dy$$

$$= 2\int_0^4 2\sqrt{y}\,dy = 4\left[\frac{y^{3/2}}{3/2}\right]_0^4$$

$$= 4\left[\frac{(4)^{3/2}}{3/2} - 0\right] = 4 \times \frac{2}{3} \times 4 \times 2$$

$$= \frac{64}{3} \text{ square unit}$$

NCERT QUESTIONS

Q. 1. Sketch the graph of $y = |x + 3|$ and evaluate $\int_6^0 |x + 3|\,dx$.

Solution :

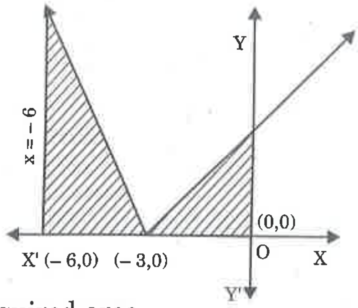
Given $\quad y = |x + 3| = \begin{cases} x+3, & x+3 \geq 0 \\ -(x+3), & x+3 < 0 \end{cases}$

$\Rightarrow \quad y = \begin{cases} x+3, & x \geq -3 \\ -x-3, & x < -3 \end{cases}$

Thus $\quad y = x + 3$, for $x \geq -3$

and $\quad y = -x - 3$, for $x < -3$

Clearly $y = x + 3$ is a straight line through $(-3, 0)$ and $(0, 3)$ which lie on x-axis and y-axis respectively. Similarly $y = -x - 3$ also a straight line.



Hence, required area

$$= \int_{-6}^{0} |x+3|\,dx$$

$$= \int_{-6}^{-3} |x+3|\,dx + \int_{-3}^{0} |x+3|\,dx$$

$$= \int_{-6}^{-3} (-x-3)\,dx + \int_{-3}^{0} (x+3)\,dx$$

$$= -\left(\frac{x^2}{2} + 3x\right)_{-6}^{-3} + \left(\frac{x^2}{2} + 3x\right)_{-3}^{0}$$

$$= -\left(\frac{9}{2} - 9 - \frac{36}{2} + 18\right) + \left(0 - \frac{9}{2} + 9\right)$$

$$= -\left(\frac{9}{2} - 18 + 18 - 9\right) + \left(\frac{9}{2}\right)$$

$$= \frac{9}{2} + \frac{9}{2} = 9 \text{ sq. units}$$

Q. 2. Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.

Solution :

Given $\quad y^2 = 4ax \quad$ ...(1)

and $\quad y = mx \quad$ ...(2)

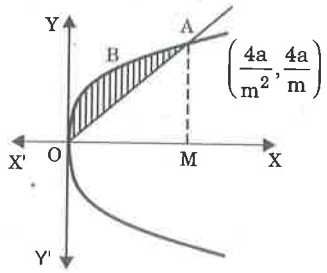
Clearly $y^2 = 4ax$ is a parabola which passes through origin and line $y = mx$ is a line passing through origin. Solving equations (1) and (2), we get the points of intersection as (0, 0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

Required area

= Area of OBAMO – Area of OAMO

$$= \int_0^{4a/m^2} (\text{for parobola } y)\,dx - \int_0^{4a/m^2} (\text{for line } y)\,dx$$

$$= \int_0^{4a/m^2} 2\sqrt{ax}\,dx - \int_0^{4a/m^2} mx\,dx$$



$$= 2\sqrt{a}\,\frac{2}{3}\left[(x)^{3/2}\right]_0^{4a/m^2} - \left[\frac{mx^2}{2}\right]_0^{4a/m^2}$$

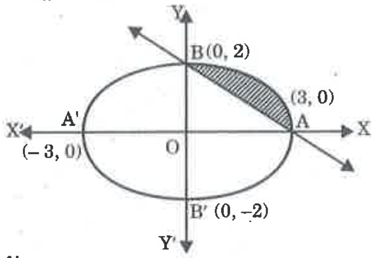
$$= \left[\frac{4\sqrt{a}}{3}\cdot\frac{8}{m^3}\cdot a^{3/2} - \frac{m}{2}\,\frac{16a^2}{m^4}\right]$$

$$= \left[\frac{32a^2}{3m^3} - \frac{8a^2}{m^3}\right] = \left(\frac{8a^2}{3m^3}\right) \text{ sq. units}$$

Q. 3. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution

We have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)



and $\frac{x}{a} + \frac{y}{b} = 1$ ...(2)

Now, required area

$$= \int_0^a (\text{for an ellipse } y)\,dx - \int_0^a (\text{for a line } y)\,dx$$

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx - \int_0^a \frac{b(a-x)}{a}dx$$

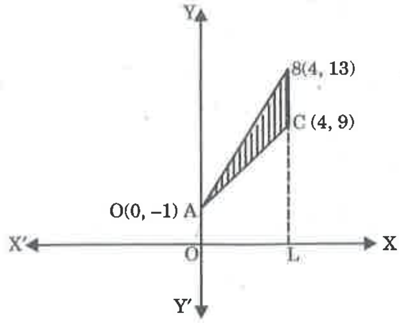
$$= \frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a - \frac{b}{a}\left[ax - \frac{x^2}{2}\right]_0^a$$

$$= \frac{ab}{a}(\sin^{-1}1 - \sin^{-1}0) - \left(ab - \frac{ab}{2}\right)$$

$$= \left(\frac{\pi ab}{4} - \frac{ab}{2}\right) = \frac{ab}{4}(\pi - 2) \text{ sq. units.}$$

Q. 4. Using integration, find the area of the region bounded by the triangle whose sides are $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. (*Raj. Board*, 2014)

Solution :



Given, $y = 2x + 1$ ...(1)

$y = 3x + 1$ ...(2)

$x = 4$ ...(3)

From equations (1) and (2), $x = 0, y = 1$

From equations (2) and (3), $x = 4, y = 13$

From equations (3) and (1),

$x = 4, y = 9$

Hence, required area

= Area of ABLOA – Area of AOLCA

$$= \int_0^4 y\,dx\,((\text{for AB}) - \int_0^4 y\,dx\,(\text{for AC})$$

$$= \int_0^4 (3x + 1)\,dx - \int_0^4 (3x+1)\,dx$$

$$= \int_0^4 (x)\,dx = \left[\frac{x^2}{2}\right]_0^4 = [8 - 0] = 8 \text{ sq. units}$$

■■

10 DIFFERENTIAL EQUATIONS

IMPORTANT FORMULAE

1. A differential equation is an equation that involves the independent variable x, the dependent variable y and the derivatives of the dependent variable w.r.t. the independent triangle. For example :
$$\frac{2d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$
2. The order of a differential equation is the order of the highest order derivative occurring in it.
3. The degree of a differential equation is the power of the highest order derivative occurring in the given differential equation.
4. Any relation between dependent and independent variables which when substituted in the differential equation reduce it to an identity is called a solution of the differential equation.
5. **General Solution :** The solution of an ordinary differential equation of nth order which contains n arbitrary constants is called the general solution of that differential equation.
6. If the differential equation has functions of x and y both, we have to separate each variable along with their differentials. This method is called variable separable form.
7. A differential equation in x and y is said to be homogeneous, if it can be put in the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of same degree in x and y.
8. An equation of the form $\frac{dy}{dx}$ + Py = Q, when P and Q are functions of x only is called a linear differential equation of first order with y as the dependent variable to be solve for such an equation.

Multiple Choice Questions

1. The differential equation
$$e^x \frac{dy}{dx} = 3y^3$$
can be solved using the method of
(a) separating the variables

(b) homogeneous equations
(c) linear differential equation of first order
(d) none of these

2. The differential equation
$$(x^3 + y^3)\, dy - x^2 y\, dx = 0$$
can be solved by using the method of
(a) separating the variables
(b) homogeneous equations
(c) linear differential equation of first order
(d) none of these

3. The differential equation
$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$$
can be solved by using the method of
(a) separating the variables
(b) homogeneous equations
(c) linear differential equation of first order
(d) none of these

4. Integrating factor of the differential equation
$$\frac{dy}{dx} + Py = Q$$
where P and Q are functions of x_1 is :
(a) $e^{\int P\,dx}$ (b) $\int e^{P\,dx}$
(c) $e^{-\int P\,dx}$ (d) none of these

5. The differential equation corresponding to the curve $y = a \sin px + b \cos px$ is (*BSEB*, 2010)
(a) $y'' + py = 0$ (b) $y'' + p^2 y = 0$
(c) $y'' - py = 0$ (d) $y'' + p^2 y = 0$

6. Which one of the following is the order of the differential equation (*BSEB*, 2011)
$$\frac{d^2y}{dx^2} + x^3\left(\frac{dy}{dx}\right) = x^4 ?$$
(a) 1 (b) 2
(c) 3 (d) none of these

7. The degree of the differential equation
$$\left(\frac{d^3y}{dx^3}\right)^2 + \frac{dy}{dx} + \cos\left(\frac{dy}{dx}\right) + 7 = 0 \text{ is}$$
(a) 2 (b) 1 (c) 3 (d) not defined

8. The proper substitution to solve the differential equation
$$(x - y)\, dy - (x + y)\, dx = 0 \text{ is}$$
(a) $y = vx$ (b) $y = \frac{v}{x}$ (c) $y = vx^2$ (d) $y = v^2 x$

9. The integrating factor of the differential equation
$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x} \text{ is}$$
(a) $\tan^{-1} x$ (b) $e^{\tan^{-1} x}$
(c) $e^{-\tan^{-1} x}$ (d) none of these

10. The differential equation for different values of A and B in the curve $y = A e^x + B e^{-x}$ is
(a) $\frac{d^2y}{dx^2} - 2y = 0$ (b) $\frac{d^2y}{dx^2} = y$
(c) $\frac{d^2y}{dx^2} = 4y + 3$ (d) $\frac{d^2y}{dx^2} + y = 0$

11. The solution of the differential equation $\frac{dy}{dx} = \frac{x}{y}$ is : (*BSEB*, 2015)
(a) $x - y = k$ (b) $x^2 - y^2 = k$
(c) $x^3 - y^3 = k$ (d) $xy = k$

12. The integrating factor of the linear differential equation $\frac{dy}{dx} + y \sec^2 x = \tan x \sec^2 x$ is : (*BSEB*, 2015)
(a) $\tan x$ (b) $e^{\tan x}$ (c) $\log \tan x$ (d) $\tan^2 x$

13. The degree of the differential equation $1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2}\right)$ is : (*BSEB*, 2015)
(a) 1 (b) 2 (c) 3 (d) 4

14. The differential equation $\frac{d^2y}{dx^2} + x^3\left(\frac{dy}{dx}\right)^3 = x^4$ is : (*BSEB*, 2015)
(a) 1 (b) 2 (c) 3 (d) 4

Ans. 1. (a), 2. (b), 3. (c), 4. (a), 5. (b), 6. (b), 7. (d), 8. (a), 9. (b), 10. (b), 11. (b), 12. (b), 13. (a), 14. (b).

Very Short Answer Type Questions

Q. 1. Write the degree of the differential equation :
$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0 \quad (CBSE, 2013)$$

Solution :

Highest order derivative present in the differential equation $= \frac{d^2y}{dx^2}$

Its power = 2

∴ Degree of the differential equation = 2

Q. 2. Write the degree of the differential equation :
$$x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0 \quad (CBSE, 2013)$$

Solution :

Highest order derivative present in the differential equation $= \frac{d^2y}{dx^2}$

Its power = 3

∴ Degree of the differential equation = 3

Q. 3. Write the degree of the differential equation :
$$\left(\frac{d^2s}{dt^2}\right)^2 + y\left(\frac{ds}{dt}\right)^3 + 4 = 0$$
[*CBSE*, 2013 (*Comptt.*)]

Solution :

Highest order derivative present in the differential equation $= \frac{d^2S}{dt^2}$

Its power = 2

$\therefore$ Degree of the differential equation = 2

Q. 4. Write the differential equation representing the family of curve $y = mx$, where m is an arbitrary constant. (*AI CBSE*, 2013)

Solution :

The equation of the family of curve is

$$y = mx \quad \ldots(1)$$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = m \quad \ldots(2)$$

Eliminating m between (1) and (2), we get

$$\frac{dy}{dx} = \frac{y}{x}$$

which is the required differential equation.

Q. 5. Verify that $y = e^x (\sin x + \cos x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \quad (JAC, 2014)$$

Solution :

$$y = e^x (\sin x + \cos x) \quad \ldots(1)$$

$$\Rightarrow \frac{dy}{dx} = e^x (\sin x + \cos x) + e^x (\cos x - \sin x)$$

$$\Rightarrow \frac{dy}{dx} = y + e^x (\cos x - \sin x) \quad \text{[from (1)]} \quad \ldots(2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x (\cos x - \sin x) + e^x (-\sin x - \cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x (\cos x - \sin x) - e^x (\sin x + \cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y \quad \text{[from (1) and (2)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Q. 6. Solve : $x\,dx + y\,dy = x\,dy - y\,dx$ (*JAC*, 2013)

Solution :

$$x\,dx + y\,dy = x\,dy - y\,dx$$

$$\Rightarrow \frac{x\,dx + y\,dy}{x^2 + y^2} = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$$\Rightarrow \frac{1}{2}\left(\frac{2x\,dx + 2y\,dy}{x^2 + y^2}\right) = \frac{d(y/x)}{1 + (y/x)^2}$$

$$\Rightarrow \frac{1}{2}\,d\log(x^2 + y^2) = d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

Integration yields

$$\frac{1}{2}(x^2 + y^2) = \tan^{-1}\left(\frac{y}{x}\right) + C,$$

where C is an arbitrary constant of integration.

Q. 7. Solve : $y\,dx - x\,dy = xy\,dx$ (*JAC*, 2013)

Solution :

$$y\,dx - x\,dy = xy\,dx$$

$$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = dx$$

Integration yields

$$\log x - \log y = x + \log C$$

where log C is an arbitrary constant of integration

$$\Rightarrow \log\left(\frac{x}{Cy}\right) = x$$

$$\Rightarrow x = Cy\,e^x,$$ which is the required solution.

Q. 8. Find the general solution of differential equation

$$\frac{dy}{dx} + y = 1 \ (y \neq 1) \quad (BSER, 2014)$$

Solution :

$$\frac{dy}{dx} + y = 1$$

$$\Rightarrow \frac{dy}{dx} = 1 - y$$

$$\Rightarrow \frac{dy}{1 - y} = dx \text{ (separating the variables)}$$

Integration yields

$$-\log(1 - y) = x + \log C$$

where log C is an arbitrary constant of integration

$$\Rightarrow \log c + \log(1 - y) = -x$$

$$\Rightarrow \log\{c(1 - y)\} = -x$$

$$\Rightarrow c(1 - y) = -e^{-x}$$

$$\Rightarrow 1 - y = \frac{1}{c}e^{-x}$$

$$\Rightarrow 1 - y = c_1 e^{-x}$$

$$\Rightarrow 1 - y = c_1 e^{-x} \text{ where } c_1 = \frac{1}{c}$$

Q. 9. Write the degree of the differential equation :

$$\left(\frac{dy}{dx}\right)^4 3x\frac{d^2y}{dx^2} = 0 \quad (CBSE, 2013)$$

Solution :

Highest order derivative present in the differential equation = $\frac{d^2y}{dx^2}$

Its power = 1

$\therefore$ Degree of the differential equation = 1

Q. 10. Write down the order and degree of the equation :

$$8x^2\frac{d^2y}{dx^2} - 7\left(\frac{dy}{dx}\right)^2 + 9 = 0 \quad (BSEB, 2014)$$

Solution :

Highest derivative present in the differential equation = $\frac{d^2y}{dx^2}$

It order = 2

Its power = 1

$\therefore$ Order and degree of the given differential equation are 2 and 1 respectively.

Q. 11. Write true or false :

The integrating factor of the equation $\frac{dy}{dx} + 2y \tan x = \sin x$ is $\tan^2 x$. (*BSEB*, 2014)

Solution :

$$P = 2\tan x$$

$$\therefore \quad IF = e^{\int P\,dx} = e^{\int 2\tan x\,dx} = e^{2\log\sec x} = e^{\log\sec^2 x} = \sec^2 x \neq \tan^2 x$$

Hence the given statement is false.

Short Answer Type Questions

Q. 1. Find the general solution of the differential equation :

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0;\ x \neq 1$$

(*BSER*, 2013)

Solution :

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$$

$$\Rightarrow \quad \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

Integration yields

$\sin^{-1} y + \sin^{-1} x = \sin^{-1} C$, where $\sin^{-1} C$ is an arbitrary constant of integration.

$$\Rightarrow \quad \sin^{-1}\left\{y\sqrt{1-x^2} + x\sqrt{1-y^2}\right\} = \sin^{-1} C$$

$$\Rightarrow \quad y\sqrt{1-x^2} + x\sqrt{1-y^2} = C,$$

which is the required solution of the given differential equation.

Q. 2. Solve the differential equation :

$$x\frac{dy}{dx} = x + y$$

(*USEB*, 2014)

Solution :

$$x\frac{dy}{dx} = x + y$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 1$$

$$P = -\frac{1}{x},\ Q = 1$$

$$IF = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x = \frac{1}{x}}$$

$\therefore$ Solution is

$$y\left(\frac{1}{x}\right) = \int 1.\frac{1}{x}dx + \log c,$$

where $\log c$ is an arbitrary constant of integration

$$\Rightarrow \quad \frac{y}{x} = \log x + \log C$$

$$\Rightarrow \quad \frac{y}{x} = \log(xc)$$

Q. 3. Find the general solution of the differential equation :

$$x\frac{dy}{dx} + 2y = x^2 (x \neq 0)$$

(*USEB*, 2014)

Solution :

$$x\frac{dy}{dx} + 2y = x^2$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = x$$

$$P = \frac{2}{x}$$

$$Q = x$$

and $$IF = e^{\int P\,dx} = e^{\int 2/x\,dx} = e^{2\log x = x^2}$$

$\therefore$ Solution is

$$yx^2 = \int x\,x^2 dx + C = \int x^3 dx + C = \frac{x^4}{4} + C$$

Q. 4. Find the general solution of the differential equation :

$$y\,dx - (x + 2y^2)\,dy = 0$$

(*USEB*, 2013)

Solution :

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x+2y^2}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{x}{y} = 2y$$

$$P = -\frac{1}{y},\ Q = 2y$$

$$IF = e^{\int P\,dy} = e^{-\int 1/y\,dy} = \frac{1}{y}$$

$\therefore$ Solution is

$$x\frac{1}{y} = \int 2y\frac{1}{y}dy + C$$

$$\Rightarrow \quad \frac{x}{y} = 2y + C$$

Q. 5. Find the particular solution of the differential equation :

$$\frac{dy}{dx} = 1 + x + y + xy, \text{ given that}$$

$y = 0$ when $x = 1$. (*AI CBSE*, 2014)

Solution :

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x)(1+y)$$

$$\Rightarrow \quad \frac{dy}{1+y} = (1+x)\,dx$$

(Separating the variables)

Integration yields

$$\log(1+y) = x + \frac{x^2}{2} + C$$

where $$x = 1,\ y = 0$$

$$\therefore \quad 0 = 1 + \frac{1}{2} + C$$

$$\Rightarrow \quad 3v + 3y^2 = y\frac{dv}{dy}$$

$$\Rightarrow \quad \frac{dv}{dy} = \frac{3}{y}v + 3y$$

$$P = -\frac{3}{y},\ Q = 3y$$

$$IF = e^{\int P\,dy} = e^{-\int 3/y\,dy} = e^{-3\log y} = \frac{1}{y^3}$$

∴ Solution is

$$v\frac{1}{y^3} = \int 3y\frac{1}{y^3}dy + C$$

$$= 3\int\frac{1}{y^2}dy + C$$

$$\Rightarrow \quad \frac{x^3}{y^3} = -\frac{3}{y} + C$$

$$\Rightarrow \quad x^3 = -3y^2 + Cy^3$$

$$\Rightarrow \quad x^3 + 3y^2 = Cy^3$$

Q. 11. Solve the following differential equation :

$$x\frac{dy}{dx} = y - x\tan\frac{y}{x} \qquad \text{(JAC, 2009, 14)}$$

Solution :

Given :

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right) \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \quad ...(1)$$

Taking $\frac{y}{x} = v \Rightarrow y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v.1 + x\frac{dv}{dx} \quad ...(2)$$

From equations (1) and (2),

$$v + x\frac{dv}{dx} = v - \tan v \quad \Rightarrow x\frac{dv}{dx} = -\tan v$$

$$\Rightarrow \quad \cot v + \frac{dx}{x} = 0$$

Integrating on both sides,

$$\Rightarrow \quad \log\sin v + \log x = \log C$$

$$\Rightarrow \quad \log(\sin v.x) = \log C$$

$$\Rightarrow \quad x\sin v = C$$

$$\Rightarrow \quad x\sin\left(\frac{y}{x}\right) = C$$

It is the required solution.

Q. 12. Solve the following differential equation :

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1} \qquad \text{(CBSE, 2014)}$$

Solution :

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{2}{(x^2-1)^2}$$

$$P = \frac{2x}{x^2-1},\ Q = \frac{2}{(x^2-1)^2}$$

$$IF = e^{\int P\,dx}$$

$$= e^{\int\frac{2x}{x^2-1}dx}$$

$$= e^{\log(x^2-1)}$$

$$= x^2 - 1$$

∴ Solution is

$$y(x^2-1) = \int\frac{2}{(x^2-1)^2}(x^2-1)\,dx + C$$

$$= 2\int\frac{1}{x^2-1}dx + C$$

$$= 2.\frac{1}{2}\log\frac{x-1}{x+1} + C$$

$$= \log\frac{x-1}{x+1} + C$$

Q. 13. Find the particular solution of the differential equation :

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

given that $y = \frac{\pi}{2}$, when $x = 1$. (CBSE, 2014)

Solution : $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

$$\Rightarrow \quad \left(\sin y + \underset{I}{y}\ \underset{II}{\cos y}\right)dy = \left(2\underset{I}{x}\ \underset{II}{\log x} + x\right)dx$$

Integrating, we get

$$-\cos y + y\sin y - \int 1.\sin y\,dy = \left[\log x.\frac{x^2}{2} - \int\frac{1}{x}.\frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y = x^2\log x - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad y\sin y = x^2\log x + C$$

when $x = 1, y = \frac{\pi}{2}$

$$\therefore \quad \frac{\pi}{2} = 0 + C \Rightarrow C = \frac{\pi}{2}$$

$$\therefore \quad y\sin y = x^2\log x + \frac{\pi}{2}$$

This gives the required particular solution.

Q. 14. Find the particular solution of the differential equation :

$$x\frac{dy}{dx} - y + x\,\text{cosec}\left(\frac{y}{x}\right) = 0;$$

given that $y = 0$ when $x = 1$

[AI CBSE, 2014 (Comptt.)]

Solution :

$$x\frac{dy}{dx} - y + x\,\text{cosec}\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x\,\text{cosec}\left(\frac{y}{x}\right)}{x}$$

Put $y = vx$

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{vx - x\,\text{cosec}\, v}{x}$$
$$= v - \text{cosec}\, v$$
$$\Rightarrow \quad x\frac{dv}{dx} = -\text{cosec}\, v$$
$$\Rightarrow \quad \sin v\, dv = -\frac{dx}{x}$$

Integration yields

$$-\cos v = -\log x - C$$
$$\Rightarrow \quad \cos v = \log x + C$$
$$\Rightarrow \quad \cos\left(\frac{y}{x}\right) = \log x + C$$

when $x = 1; y = 0$

$$\therefore \quad 1 = 0 + C \Rightarrow C = 1$$
$$\therefore \quad \cos\left(\frac{y}{x}\right) = \log x + 1$$

Q. 15. Solve the following differential equation :

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x; x \neq 0.$$

[AI CBSE, 2014 (Comptt.)]

Solution :

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

Put $y = vx$

so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{vx\cos v + x}{x\cos v}$$
$$= \frac{v\cos v + 1}{\cos v}$$
$$= vx \sec v$$
$$\Rightarrow \quad x\frac{dv}{dx} = \sec v$$
$$\Rightarrow \quad \cos v\, dv = \frac{dx}{x}$$

Integration yields

$$\sin v = \log x + C$$
$$\Rightarrow \quad \sin\frac{y}{x} = \log x + C$$

Q. 16. Find the particular solution of the differential equation

$$\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0; \text{ given that}$$

$y = 0$ when $x = 1$. *(BSEB, 2014)*

Solution :

$$\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0$$

Put $y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad vx\frac{dv}{dx} - v + \text{cosec}\, v = 0$$
$$\Rightarrow \quad x\frac{dv}{dx} + \frac{1}{\sin v} = 0$$
$$\Rightarrow \quad \sin v\, dx + \frac{dx}{x} = 0$$

Integration yields

$$-\cos v + \log x = \log C$$
$$\Rightarrow \quad \log x - \log C = \cos v$$
$$\Rightarrow \quad \log\left(\frac{x}{C}\right) = \cos\left(\frac{y}{x}\right)$$
$$\Rightarrow \quad \frac{x}{C} = e^{\cos\left(\frac{y}{x}\right)}$$
$$\Rightarrow \quad x = Ce^{\cos\left(\frac{y}{x}\right)}$$

when $x = 1, y = 0$

$$\therefore \quad 1 = Ce' \Rightarrow C = \frac{1}{e}$$
$$\therefore \quad x = \frac{1}{e}e^{\cos\left(\frac{y}{x}\right)}$$

This gives the required particular solution.

Q. 17. Find a particular solution of the differential equation :

$$(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx,$$

given that $y = 0$ if $x = \frac{\pi}{2}$. *(BSER, 2013)*

Solution :

$$(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$$
$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2}$$
$$P = \frac{2x}{1+x^2}$$

and

$$Q = \frac{\cot x}{1+x^2}$$
$$IF = e^{\int P\,dx}$$
$$= e^{\int\frac{2x}{1+x^2}dx}$$
$$= e^{\log(1+x^2)}$$
$$= 1 + x^2$$

$\therefore$ Solution in

$$y(1 + x^2) = \int\frac{\cot x}{1+x^2}(1+x^2)dx + C$$
$$= \int\cot x\, dx + C$$
$$= \log\sin x + C$$

when $x = \frac{\pi}{2}, y = 0$

$$\therefore \quad 0 = 0 + C \Rightarrow C = 0$$
$$\therefore \quad y(1 + x^2) = \log\sin x$$

This gives the required particular solution

Q. 18. Solve the differential equation :

$$(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$ *(AI CBSE, 2014)*

Solution :

$$(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$$

$$P = \frac{1}{1+x^2},\ Q = \frac{e^{\tan^{-1}x}}{1+x^2}$$

$$IF = e^{\int P\,dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$$

∴ Solution is

$$y\,e^{\tan^{-1}x} = \int e^{\tan^{-1}x}\frac{e^{\tan^{-1}x}}{1+x^2}dx + C$$

Put $e^{\tan^{-1}x} = t$

then $\frac{e^{\tan^{-1}x}}{1+x^2}dx = dt$

$$= \int t\,dt + C$$

$$= \frac{t^2}{2} + C$$

$$= \left(\frac{e^{\tan^{-1}}x}{2}\right)^2 + C$$

$$\Rightarrow \quad y = \frac{1}{2}e^{\tan^{-1}x} + C\,e^{-\tan^{-1}x}$$

Q. 19. Find the particular solution of the differential equation $x(1+y^2)\,dx - y(1+x^2)\,dy = 0$, given that $y = 1$ when $x = 0$. *(AI CBSE, 2014)*

Solution :

$$x(1+y^2)\,dx - y(1+x^2)\,dy = 0$$

$$\Rightarrow \quad \frac{x\,dx}{1+x^2} = \frac{y\,dx}{1+y^2}$$

$$\Rightarrow \quad \frac{2x\,dx}{1+x^2} = \frac{2y\,dy}{1+y^2}$$

Integration yields

$$\log(1+x^2) = \log(1+y^2) + C$$

when $x = 0, y = 1$

$\therefore\ 0 = \log 2 + C \Rightarrow C = -\log 2$

$$\therefore \quad \log(1+x^2) = +\log(1+y^2) - \log 2$$

$$\Rightarrow \log(1+y^2) - \log(1+x^2) = \log 2$$

$$\Rightarrow \quad \log\frac{1+y^2}{1+x^2} = \log 2$$

$$\Rightarrow \quad \frac{1+y^2}{1+x^2} = 2$$

$$\Rightarrow \quad 1+y^2 = 2 + 2x^2$$

$$\Rightarrow \quad y^2 - 2x^2 - 1 = 0$$

This gives the required particular solution.

Q. 20. Find the particular solution of the differential equation :

$\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$, given that $y = 1$ when $x = 0$.

(CBSE, 2014)

Solution :

$$e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$$

$$\Rightarrow \quad \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\,dx + \frac{y}{\sqrt{1-y^2}}dy = 0$$

Integration yields

$$x\,e^x - \int e^x dx - \frac{1}{2}\int\frac{-2y}{\sqrt{1-y^2}}dy = 0$$

Put $1 - y^2 = t^2$

$$\Rightarrow \quad -2y\,dy = 2t\,dt$$

$$\Rightarrow \quad x\,e^x - e^x - \frac{1}{2}\int\frac{2t\,dt}{t} = C$$

$$\Rightarrow \quad x\,e^x - e^x - t = C$$

$$\Rightarrow \quad x\,e^x - e^x - \sqrt{1-y^2} = C$$

when $x = 0, y = 1$

$\therefore\ 0 - 1 - 0 = C \Rightarrow C = -1$

$$\therefore \quad x\,e^x - e^x - \sqrt{1-y^2} = -1$$

$$\Rightarrow \quad \sqrt{1-y^2} = x\,e^x - e^x + 1$$

This gives the required particular solution.

Q. 21. Solve the following differentiate equation :

$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0$ *(CBSE, 20 4)*

Solution :

$$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0$$

$$\Rightarrow \quad \frac{\log y}{y^2}dy + x^2 \sin x\,dx = 0$$

Integration yields

$$\int \underset{\text{I}}{\log y}.\underset{\text{II}}{\frac{1}{y^2}}dy + \int \underset{\text{I}}{x^2}\,\underset{\text{II}}{\sin x}\,dx = C$$

$$\Rightarrow \log y\left(\frac{-1}{y}\right) - \int\frac{1}{y}\left(-\frac{1}{y}\right)dy + x^2(-\cos x) - \int 2x(-\cos x)dx = C$$

$$\Rightarrow \quad -\frac{\log y}{y} - \frac{1}{y} - x^2\cos x + 2\int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos x}\,dx = C$$

$$\Rightarrow \quad -\frac{\log y}{y} - \frac{1}{y} - x^2\cos x + 2\left[x\sin x - \int 1.\sin x\,dx\right] = C$$

$$\Rightarrow \quad -\frac{\log y}{y} - \frac{1}{y} - x^2\cos x + 2x\sin x + 2\cos x = C$$

$$\Rightarrow \quad -\frac{1+\log y}{y} = x^2\cos x - 2(x\sin x + \cos x) + C$$

Long Answer Type Questions

Q. 1. Solve the differential equation :

$(x - y)\,dy - (x + y)\,dx = 0$ *(USEB, 2013)*

Solution :

$$(x-y)\,dy - (x+y)\,dx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y}{x-y}$$

Put $y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$= \frac{1+v-v+v^2}{1-v}$$

$$= \frac{1+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1-v^2}\,dv = \frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{1+v^2} - \frac{1}{2}\frac{2v}{1+v^2}\right)dv = \frac{dx}{x}$$

Integration yields

$$\tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + \log C$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(1+\frac{y^2}{x^2}\right) = \log x + \log C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} - \frac{1}{2}\log(x^2+y^2) + \log x = \log x + \log C$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x} = \frac{1}{2}\log(x^2+y^2) + \log C$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x} = \log\{C\sqrt{x^2+y^2}\}$$

$$\Rightarrow \quad C\sqrt{x^2+y^2} = e^{\tan^{-1}\frac{y}{x}}$$

Q. 2. Show that the differential equation $2y\, e^{x/y}\, dx + (y - 2x\, e^{x/y})\, dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x = 0$ when $y = 1$. (*CBSE*, 2013)

Solution :

$$2y\, e^{x/y}\, dx + (y - 2x\, e^{x/y})\, dy = 0 \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{2x\,e^{x/y} - y}{2y\,e^{x/y}} = f(x, y) \text{ say}$$

then

$$f(\lambda x, \lambda y) = \frac{2\lambda x\, e^{\frac{\lambda x}{\lambda y}} - \lambda y}{2\lambda y\, e^{\frac{\lambda x}{\lambda y}}}$$

$$= \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}} = f(x, y)$$

∴ (1) is homogenous.

Put $\quad x = vy$

So that $\quad \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

then (1) reduces to

$$v + y\frac{dv}{dy} = \frac{2vy\,e^{vy/y} - y}{2y\,e^{vy/y}}$$

$$= \frac{2ve^v - 1}{2e^v}$$

$$= v - \frac{1}{2e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{1}{2e^v}$$

$$\Rightarrow \quad 2\,e^v\,dv = -\frac{dy}{y}$$

Integration yields

$$2e^v = -\log y + C$$

$$\Rightarrow \quad 2e^{x/y} = -\log y + C$$

when $y = 1, x = 0$

$$\therefore \quad 2 = 0 + C \Rightarrow C = 2$$

$$\therefore \quad 2e^{x/y} = -\log y + 2$$

$$\Rightarrow \quad 2e^{x/y} + \log y = 2$$

This gives the required particular solution.

Q. 3. Show that the differential equation $(x\, e^{y/x} + y)\, dx = x\, dy$ is homogeneous. Find the particular solution of this differential equation, given that $x = 1$ when $y = 1$. (*CBSE*, 2013)

Solution :

$$(x\,e^{y/x} + y)\,dx = x\,dy \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xe^{y/x} + y}{x}$$

Let $\quad f(x, y) = \dfrac{xe^{y/x} + y}{x}$,

then $\quad f(\lambda x, \lambda y) = \dfrac{\lambda x e^{\frac{\lambda y}{\lambda x}} + \lambda y}{\lambda x}$

$$= \frac{xe^{y/x} + y}{x}$$

$$= f(x, y)$$

∴ (1) is homogeneous.

Put $\quad y = vx$

so that $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

∴ (1) reduces to

$$v + x\frac{dv}{dx} = \frac{xe^{vx/x} + vx}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = e^v + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = e^v$$

$$\Rightarrow \quad e^{-v}\,dv = \frac{dx}{x}$$

Integration yields

$$-e^{-v} = \log x + C$$

$$\Rightarrow \quad -e^{-y/x} = \log x + C$$

when $x = 1, y = 1$

$$\therefore \quad -e^{-1} = \log 1 + C$$

$$\Rightarrow \quad -e^{-1} = 0 + C$$

$$\Rightarrow \quad C = -e^{-1}$$

$$\therefore \quad -e^{-y/x} = \log x - e^{-1}$$

$$\Rightarrow \quad e^{-1} - e^{-y/x} = \log x$$

This gives the required particular solution.

Q. 4. Show that the differential equation :

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right) = 0 \text{ is homogeneous.}$$

Find the particular solution of this differential equation, given that $x = 1$ when $y = \frac{\pi}{2}$. (*CBSE*, 2013)

Solution :

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right)-x}{x\sin\left(\frac{y}{x}\right)} = f(x, y) \text{ say} \quad ...(1)$$

then $$f(\lambda x_1, \lambda y) = \frac{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x}{\lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}$$

$$= \frac{y\sin\left(\frac{y}{x}\right)-x}{x\sin\left(\frac{y}{x}\right)}$$

$$= f(x, y)$$

$\therefore$ (1) is homogeneous.

Put $y = vx$

then $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{vx\sin v - x}{x\sin v}$$

$$= \frac{v\sin v - 1}{\sin v}$$

$$= v - \frac{1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Rightarrow \quad -\sin v\, dv = \frac{dx}{x}$$

Integration yields

$$\cos v = \log x + C$$

$$\Rightarrow \quad \cos\frac{y}{x} = \log x + C$$

$x = 1$ when $y = \frac{\pi}{2}$

$$\therefore \quad \cos\frac{\pi}{2} = \log 1 + C$$

$$\Rightarrow \quad 0 = 0 + C \Rightarrow C = 0$$

$$\therefore \quad \cos\frac{y}{x} = \log x$$

This gives the required particular solution.

Q. 5. Find the particular solution of the differential equation :

$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$, given that when $x = 0$, $y = 0$. (*AI CBSE*, 2013)

Solution :

$$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} x}{1+y^2}$$

$$P = \frac{1}{1+y^2},\ Q = \frac{\tan^{-1} y}{1+y^2}$$

$$IF = e^{\int P\,dy} = e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

$\therefore$ Solution is

$$x\,e^{\tan^{-1} y} = \int e^{\tan^{-1} y}\frac{\tan^{-1} y}{1+y^2}\,dy + C$$

Put $\tan^{-1} y = t$

$$\therefore \quad \frac{1}{1+y^2}dy = dt$$

$$= \int \underset{\text{II}}{e^t}\,\underset{\text{I}}{t}\,dt + C$$

$$= t\,e^t - \int 1.e^t dt + C$$

$$= t\,e^t - e^t + C$$

$$= \tan^{-1} y\, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$$

$$\Rightarrow \quad x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

when $x = 0, y = 0$

$$\therefore \quad 0 = 0 - 1 + C$$

$$\Rightarrow \quad C = 1$$

$$\therefore \quad x = \tan^{-1} y - 1 + e^{-\tan^{-1} y}$$

$$\Rightarrow \quad x - \tan^{-1} y + 1 = e^{-\tan^{-1} y}$$

$$\Rightarrow \quad (x - \tan^{-1} y + 1)\, e^{\tan^{-1} y} = 1$$

This gives the required particular solution.

Q. 6. Show that the differential equation

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0 \text{ is homogeneous.}$$

Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$. (*AI CBSE*, 2013)

Solution :

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x\sin^2\left(\frac{y}{x}\right)}{x}$$

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{vx - x\sin^2 v}{x}$$

$$= v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 x$$

$\Rightarrow$ $$\int \text{cosec}^2 v\, dv = -\int \frac{dx}{x} + \log C$$

$\Rightarrow$ $$-\cot v = -\log x + \log C$$

$\Rightarrow$ $$-\cot\left(\frac{y}{x}\right) = -\log x - \log C$$

$\Rightarrow$ $$\cot\left(\frac{y}{x}\right) = \log x - \log C$$

$\Rightarrow$ $$\cot\left(\frac{y}{x}\right) = \log\left(\frac{y}{C}\right)$$

$\Rightarrow$ $$\frac{x}{C} = e^{\cot(y/x)}$$

$\Rightarrow$ $$x = Ce^{\cot(y/x)}$$

when $x = 1, y = \frac{\pi}{4}$

$\therefore$ $$1 = Ce^{\cot\frac{\pi}{4}}$$

$\Rightarrow$ $$1 = Ce^1$$

$\Rightarrow$ $$C = \frac{1}{e}$$

$\therefore$ $$x = \frac{1}{e}e^{\cot\left(\frac{y}{x}\right)}$$

This gives the required particular solution.

Q. 7. Find the particular solution of the differential equation :

$$(x-y)\frac{dy}{dx} = x + 2y,$$ **given that when $x = 1, y = 0$.**

[*CBSE*, 2013, 14 (*Comptt.*)]

Solution :

$$(x-y)\frac{dy}{dx} = x + 2y$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{x+2y}{x-y}$$

Put $$y = vx$$

$\therefore$ $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ $$v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx} = \frac{1+2v}{1-v}$$

$\Rightarrow$ $$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$= \frac{1+2v-v+v^2}{1-v}$$

$$= \frac{1+v+v^2}{1-v}$$

$\Rightarrow$ $$\frac{1-v}{1+v+v^2}dv = \frac{dx}{x}$$

$\Rightarrow$ $$\frac{v-1}{v^2+v+1}dv = -\frac{dx}{x}$$

$\Rightarrow$ $$\frac{2v-2}{v^2+v+1}dv = -\frac{2}{x}dx$$

$\Rightarrow$ $$\frac{(2v+1)-3}{v^2+v+1}dv = -\frac{2}{x}dx$$

$\Rightarrow$ $$\left[\frac{2v+1}{v^2+v+1} - \frac{3}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right]dv = -\frac{2}{x}dx$$

Integration yields

$$\log(v^2+v+1) - 3.\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{4+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = -2\log x + \log C$$

$$\Rightarrow \log\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right) - 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = 2\log x + \log C$$

$$\Rightarrow \log(y^2+xy+x^2) - 2\log x - 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = 2\log x + \log C$$

$\Rightarrow$ $$\log\left(\frac{x^2+xy+y^2}{C}\right) = 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)$$

$\Rightarrow$ $$x^2+xy+y^2 = e^{2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)}$$

when $x = 1, y = 0$

$$\therefore\ 1 = Ce^{2\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)} \Rightarrow 1 = Ce^{2\sqrt{3}\frac{\pi}{6}} \Rightarrow 1 = Ce^{\frac{\pi}{\sqrt{3}}}$$

$\therefore$ $$x^2+xy+y^2 = e^{2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)-\frac{\pi}{\sqrt{3}}}$$

This gives the required particular mention

Q. 8. Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\cot x = 2x + x^2\cot x\ (x \neq 0),$$ **given that $y = 0$, when $x = \frac{\pi}{2}$.** (*BSER*, 2014)

Solution :

$$\frac{dy}{dx} + y\cot x = 2x + x^2\cot x$$

$$P = \cot x$$

and $$Q = 2x + x^2\cot x$$

$$IF = e^{\int P\,dx}$$

$$= e^{\int \cot x\,dx}$$

$$= e^{\log \sin x}$$

$$= \sin x$$

$\therefore$ Solution is

$$y\sin x = \int(2x + x^2\cot x)\sin x\,dx + C$$

$$= 2\int x\sin x\,dx + \int \underset{I}{x^2}\ \underset{II}{\cos x}\,dx + C$$

$$= 2\int x\sin x\,dx + x^2\sin x - \int 2x\sin x\,dx + C$$

$$= x^2\sin x + C$$

when $$x = \frac{\pi}{2}, y = 0$$

$\therefore$ $$0 = \left(\frac{\pi}{2}\right)^2 - \sin\frac{\pi}{2} \Rightarrow C = -\frac{\pi^2}{4}$$

$\therefore$ $$y\sin x = x^2\sin x - \frac{\pi^2}{4}$$

Q. 9. Find the particular solution of the differential equation :

$$\frac{dy}{dx} + x\cot y = 2y + y^2 \cot y, (y \neq 0)$$

given that $x = 0$ when $y = \frac{\pi}{2}$. (*AI CBSE*, 2013)

Solution :

$$\frac{dy}{dx} + x\cot y = 2y + y^2 \cot y$$

$P = \cot y$ and $Q = 2y + y^2 \cot y$

$$IF = e^{\int P dy} = e^{\int \cot y\, dy} = e^{\log \sin y} = \sin y$$

$\therefore$ Solution is

$$x \sin y = \int (2y + y^2 \cot y)\sin y + C$$

$$= 2y\int \sin y\, dy + 2\int \underset{I}{y^2} \underset{I}{\sin} y\, dy + C$$

$$= 2\int y \sin y\, dy + y^2 \sin y - \int 2y \sin y\, dy + C$$

$$= y^2 \sin y + C$$

when $\quad y = \frac{\pi}{2}, x = 0$

$$\therefore \quad 0 = \left(\frac{\pi}{2}\right)^2 .1 + C \Rightarrow C = -\frac{\pi^2}{4}$$

$$\therefore \quad x \sin y = y^2 \sin y - \frac{\pi^2}{4}$$

This gives the required particular solution.

Q. 10. Find the particular solution of the differential equation :

$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$, given that when $x = 1, y = \frac{\pi}{4}$. [*CBSE*, 2013 (*Comptt.*)]

Solution :

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{y}{x}\right) + x \sec\left(\frac{y}{x}\right)$$

Put $\quad y = vx$

so that $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = v + x \sec v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sec v$$

$$\Rightarrow \quad \cos v\, dv = \frac{dx}{x}$$

Integration yields

$$\sin v = \log x + \log C$$

$$\Rightarrow \quad \sin\frac{y}{x} = \log x + \log C$$

when $\quad x = 1, y = \frac{\pi}{4}$

$$\therefore \quad \sin\frac{\pi}{4} = \log 1 + \log C$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = 0 + \log C$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \log C$$

$$\therefore \quad \sin\frac{y}{x} = \log x + \frac{1}{\sqrt{2}}$$

This gives the required particular solution.

Q. 11. Solve the following differential equation :

$$x\cos\left(\frac{y}{x}\right)(y\,dx + x\,dy) = y\sin\left(\frac{y}{x}\right)(x\,dy - y\,dx)$$

[*CBSE*, 2013 (*Comptt.*)]

Solution :

$$x\cos\left(\frac{y}{x}\right)(y\,dx + x\,dy) = y\sin\left(\frac{y}{x}\right)(x\,dy - y\,dx)$$

$$\Rightarrow \quad x\cos\left(\frac{y}{x}\right)\left(y + x\frac{dy}{dx}\right) = y\sin\left(\frac{y}{x}\right)\left(x\frac{dy}{dx} - y\right)$$

$$\Rightarrow \frac{dy}{dx}\left\{x\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)\right\} = y\left\{x\cos\frac{y}{x} + y\sin\frac{y}{x}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\left\{x\cos\frac{y}{x} + y\sin\frac{y}{x}\right\}}{x\left\{y\sin\frac{y}{x} - x\cos\frac{y}{x}\right\}}$$

Put $y = vx$

so that $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{v(x\cos v + vx\sin v)}{(vx\sin v - x\cos v)}$$

$$= \frac{v(\cos v + v\sin v)}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = v\left[\frac{\cos v + v\sin v}{v\sin v - \cos v} - 1\right]$$

$$= v\left[\frac{2\cos v}{v\sin v - \cos v}\right]$$

$$\Rightarrow \quad \frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}$$

$$\Rightarrow \quad \left(\tan v - \frac{1}{v}\right)dv = 2\frac{dx}{x}$$

$$\Rightarrow \quad -\log\cos v - \log v = 2\log x - \log C$$

$$\Rightarrow \quad \log\cos v + \log v = -2\log x + \log C$$

$$\Rightarrow \quad \log(v\cos v) + 2\log x = \log C$$

$$\Rightarrow \quad \log(v\cos v\, x^2) = \log C$$

$$\Rightarrow \quad \log\left\{\frac{y}{x}\cos\left(\frac{y}{x}\right)x^2\right\} = \log C$$

$$\Rightarrow \quad \log\left\{xy\cos\left(\frac{y}{x}\right)\right\} = \log C$$

$$\Rightarrow \quad xy\cos\left(\frac{y}{x}\right) = C$$

Q. 12. Solve the differential equation :
$(x - y)\,dy - (x + y)\,dx = 0$ (*BSEB*, 2015)

Solution :

$$(x - y)\,dy - (x + y)\,dx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x + y}{x - y}$$

Let $\quad y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v+v^2}{1-v} = \frac{1+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1+v^2}dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int\frac{1}{1+v^2}dv - \frac{1}{2}\int\frac{2v}{1+v^2}dv = \int\frac{1}{x}dx$$

$$\Rightarrow \quad \tan^{-1} v - \frac{1}{2}\log(v^2+1) - \log x = C$$

$$\Rightarrow \quad \tan^{-1} v - \frac{1}{2}\log(v^2+1) - \log x = C$$

Now, substituting $v = \frac{y}{x}$,

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(\frac{y^2+x^2}{x^2}\right) - \log x = C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) - \log\left(\frac{\sqrt{y^2+x^2}}{x} + x\right) = C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log(y^2+x^2) = C$$

Q. 13. Find the particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 4\cos x\ (x \neq 0)$ given that $y = 0$, when $x = \frac{\pi}{2}$ *(Raj. Board, 2015)*

Solution :

Differential equation

$$\frac{dy}{dx} + y\cot x = 2\cos x$$

Comparing with $\frac{dy}{dx} + Py = Q$

then $\quad P = \cot x$ and $Q = 2\cos x$

$$\therefore \quad \text{If} = e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$$

Thus, general solution

$$y\sin x = \int 2\sin x\cos x\,dx + C$$

$$\Rightarrow \quad y\sin x = \int \sin 2x\,dx + C$$

$$\Rightarrow \quad y\sin x = -\frac{\cos 2x}{2} + C \quad \ldots(1)$$

since given that $\quad y = 0$, when $x = \frac{\pi}{2}$

thus $x = \frac{\pi}{2}$ and $y = 0$ putting in eq. (1)

$$0\cdot\sin\frac{\pi}{2} = \frac{\cos 2\times\pi/2}{2} + C$$

$$\Rightarrow \quad C - \frac{\cos\pi}{2} = 0$$

$$\Rightarrow \quad C + \frac{1}{2} = 0$$

$$\Rightarrow \quad C = -\frac{1}{2}$$

Hence, particular solution is

$$y\sin x = -\frac{\cos 2x}{2} - \frac{1}{2}$$

$$\Rightarrow \quad 2y\sin x + \cos 2x + 1 = 0$$

Q. 14. Solve : $(1 + x^2)\frac{dy}{dx} + 2xy + \sqrt{x^2+4}$ *(JAC, 2015)*

Solution :

Differential equation

$$(1+x)^2\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$$

dividing by $(x^2 + 1)$

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{\sqrt{x^2+4}}{1+x^2} \quad \ldots(1)$$

Comparing equation (1) with $\frac{dy}{dx} + Py = Q$

$$P = \frac{2x}{x^2+1} \text{ and } Q = \frac{\sqrt{x^2+4}}{1+x^2}$$

$$\therefore \text{If} \quad = e^{\int\frac{2x}{x^2+1}dx} = e^{\log(x^2+1)} = x^2+1$$

Hence, general solution

$$y(x^2+1) = \int(x^2+1)\cdot\frac{\sqrt{x^2+4}}{x^2+1}dx$$

$$\Rightarrow y(x^2+1) = \int\sqrt{x^2+4}\,dx$$

$$\Rightarrow y(x^2+1) = \frac{x}{2}\sqrt{x^2+2^2} + \frac{2^2}{2}\log(x + \sqrt{x^2+2^2}) + C$$

$$\Rightarrow y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + 2\log(x + \sqrt{x^2+4}) + C$$

Q. 15. Solve : $(x^3 + y^3)\,dy - x^2y\,dx = 0$ *(JAC, 2015)*

Solution :

Differential equation

$$(x^3+y^3)\,dy - x^2y\,dx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$$

Let $\quad y = vx \Rightarrow dy = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3x^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v - v - v^4}{1+v^3} = -\frac{v^4}{1+v^3}$$

$$\Rightarrow \quad x(1+v^3)\,dv = -v^4\,dv$$

$$\Rightarrow \quad \frac{1+v^3}{v^4}dv = \frac{-dx}{x}$$

$$\Rightarrow \quad \left(\frac{1}{v^4}+\frac{1}{v}\right)dv = \frac{-dx}{x}$$

Integrating both sides

$$\frac{v^{-3}}{-3} + \log v = -\log x + C$$

$$\Rightarrow \quad -\frac{1}{3v^3} + \log v + \log x = C$$

$$\Rightarrow \quad -\frac{1}{3}\frac{x^3}{y^3} + \log\left(\frac{y}{x}\cdot x\right) = C \qquad (\because v = y/x)$$

$$\Rightarrow \quad -\frac{x^3}{3y^3} + \log y = C$$

Which is required general solution.

Q. 16. Find general solution of differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$. *(Raj. Board, 2015)*

Solution :

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\Rightarrow \quad \frac{1}{1+y^2}dy = \frac{1}{1+x^2}dx$$

Integrating both sides

$$\int\frac{1}{1+y^2}dy = \int\frac{1}{1+x^2}dx$$

$$\Rightarrow \quad \tan^{-1}y = \tan^{-1}x + \tan^{-1}C$$

$$\Rightarrow \quad \tan^{-1}y - \tan^{-1}x = \tan^{-1}C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y-x}{1+xy}\right) = \tan^{-1}C$$

$$\Rightarrow \quad \frac{y-x}{1+xy} = C$$

$$\Rightarrow \quad y - x = C(1+xy)$$

Which is required general solution.

NCERT QUESTIONS

Q. 1. Solve : $\cos^2 x\frac{dy}{dx} + y = \tan x$

(CBSE, Outside Delhi, 2011)

Solution :

We have

$$\frac{dy}{dx} + y\sec^2 x = \frac{\tan x}{\cos^2 x}$$

$$\Rightarrow \quad \frac{dy}{dx} + y\sec^2 x = \tan x.\sec^2 x$$

Now comparing with $\frac{dy}{dx} + Py = Q$,

$P = \sec^2 x$ and $Q = \tan x \sec^2 x$

thus $\quad IF = e^{\int\sec^2 x\,dx} = e^{\tan x}$

Hence the solution is :

$$y \times e^{\tan x} = \int e^{\tan x}.\tan x\sec^2 x\,dx$$

$$y\,e^{\tan x} = \int t.e^t dt$$

[Taking $\tan x = t \Rightarrow \sec^2 x\,dx = dt$]

$$\Rightarrow \quad y\,e^{\tan x} = (t.e^t - e^t) + C$$

$$\Rightarrow \quad y\,e^{\tan x} = \tan x\,e^{\tan x} - e^{\tan x} + C$$

$$\Rightarrow \quad y = (\tan x - 1) + Ce^{\tan x}.$$

Q. 2. Solve : $e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$

(CBSE, Outside Delhi, 2011)

Solution :

$$e^x \tan y\,dx = -(1-e^x)\sec^2 y\,dy$$

$$\Rightarrow \quad -\frac{e^x}{1-e^x}dx = \frac{\sec^2 y}{\tan y}dy$$

Integrate on both sides,

$$-\int\frac{e^x}{1-e^x}dx = \int\frac{\sec^2 y}{\tan y}dy$$

$$\Rightarrow \quad \log(1-e^x) = \log\tan y + \log C$$

$$\Rightarrow \quad (1-e^x) = C\tan y.$$

Q. 3. Solve : $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

[CBSE, 2011, BSER, 14]

Solution :

$$\Rightarrow \quad x\,dy = (\sqrt{x^2+y^2} + y)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{\sqrt{x^2+y^2}+y}{x}\right)$$

The degree of numerator and denominator is same, so

Put $\quad y = ux \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{\sqrt{x^2+v^2x^2}+vx}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrate on both sides,

$$\Rightarrow \quad \log\{v + \sqrt{1+v^2}\} = \log x + \log C$$

$$\Rightarrow \quad v + \sqrt{1+v^2} = Cx$$

Now taking $v = \frac{y}{x}$,

$$\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = Cx \quad \text{or} \quad y + \sqrt{x^2+y^2} = Cx^2.$$

Q. 4. Solve : $x\dfrac{dy}{dx} + 2y = x^2 \log x$

Solution :

Given, $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = x \log x$ ...(1)

Comparing with $\dfrac{dy}{dx} + Py = Q$,

$$P = \frac{2}{x} \text{ and } Q = x \log x$$

$$IF = e^{\int P\,dx} = e^{\int \frac{2}{x} dx} = e^{2 \log x}$$

$$IF = x^2$$

Now, the solution is

$$y \times x^2 = \int x^2 \times x \log x \, dx$$

$$x^2 y = \int (\underset{II}{x^3} \underset{I}{\log x})\, dx$$

$$\Rightarrow \quad x^2 y = (\log x).\frac{x^4}{4} - \int \frac{1}{x} \times \frac{x^4}{4}.dx$$

$$\Rightarrow \quad x^2 y = \frac{x^4}{4}\log x - \frac{1}{4}\int x^3 dx$$

$$\Rightarrow \quad x^2 y = \frac{x^4}{4}\log x - \frac{1}{16}x^4$$

$$\Rightarrow \quad y = \frac{x^2}{4}\log x - \frac{1}{16}x^2 + \frac{C}{x^2}.$$

Q. 5. Solve :

$(x^2 - y^2)\, dx + 2xy\, dy = 0$ (*CBSE, Delhi*, 2010)

Solution :

$$\Rightarrow \quad (x^2 - y^2)\, dx = -2xy\, dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 - y^2}{-2xy}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

The degree of numerator and denominator is 2, thus it is a homogeneous function.

Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x.(vx)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-1 - v^2}{2v}$$

$$\Rightarrow \quad \int \frac{2v}{v^2+1} dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log(v^2 + 1) = -\log x + \log C$$

$$\Rightarrow \quad \log(v^2 + 1) = \log\left(\frac{C}{x}\right)$$

Putting $v = \dfrac{y}{x}$,

$$\log\left(\frac{x^2+y^2}{x^2}\right) = \log C - \log x$$

$$\Rightarrow \quad \log(x^2 + y^2) - 2\log x = \log C - \log x$$

$$\Rightarrow \quad \log(x^2 + y^2) = \log x + \log C$$

$$\Rightarrow \quad \log(x^2 + y^2) = \log(Cx)$$

$$\Rightarrow \quad x^2 + y^2 = Cx$$

Q. 6. Solve :

$4\dfrac{dy}{dx} + 8y = 5e^{-3x}$ (*CBSE, Delhi*, 2007)

Solution :

We have $4\dfrac{dy}{dx} + 8y = 5e^{-3x}$

or $\dfrac{dy}{dx} + 2y = \dfrac{5}{4}e^{-3x}$

Hence $P = 2$ and $Q = \dfrac{5}{4}e^{-3x}$

$$\therefore \quad IF = e^{\int P\,dx} = e^{\int 2\,dx} = e^{2x}$$

Now $ye^{2x} = \displaystyle\int \frac{5}{4}e^{-3x}\, e^{2x}\, dx + C$

$$\Rightarrow \quad ye^{2x} = -\frac{5}{4}e^{-x} + C$$

$$\Rightarrow \quad y = -\frac{5}{4}e^{-3x} + Ce^{-2x}$$

■■